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Thin complete subsequence

by

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ABSTRACT

A is said to be *complete* if every sufficiently large integer belongs to sumset of A. A' is *thin comlete subsequence* of A if A' is complete and $A'(x) = (1 + o(1)) \log_2 x$.

It is proved that $\lim_{n\to\infty} a_{n+1}/a_n = 1$ implies the existence of thin complete subsequence.

1. Introduction

The well known theorem of Lagrange states that every nonnegative integer n is the sum of four squares. In other words the sequence $S = \{1, 4, \ldots, n^2, \ldots\}$ is bases of order four. Generally if A is a bases of order h then $A(x) \gg x^{1/h}$, where $A(x)$ is the counting function of A. Wirsing defined the notion of thin bases; A is *thin bases* of order h if $A(x) < c' x^{1/h}$; $(c' > 0)$. By a non-constructive method Wirsing proved [1] that S has a subbases S' which is almost thin, proving $S'(x) = O(x^{1/4}(\log x)^{1/4})$. Later J. Spencer [2] gave a short proof for it, using an important tool of probabilistic method, the Janson's inequality.

We merely mention it is not even known an explicit subsequence S' of S for which $S'(x) = o(\sqrt{x})$.

A related question would be the following: let $A = \{a_1 < \ldots < a_n < \ldots\} \subseteq \mathbf{N}$. A is said to be complete if there exists Δ_A such that for every $n \geq \Delta_A$ we have

$$n \in \Sigma(A) = \{S(B) : S(B) = \sum_{b \in B};\ B \text{ is a finite subset of } A,\ S(\emptyset) = 0\}.$$

Clearly if $|A| = k$ then $|\Sigma(A)| \leq 2^k$. This implies if A is complete then $2^{A(x)} \geq x - \Delta_A$ i.e. for $x \geq x_0$ $A(x) \geq \log_2(x - \Delta_A)$.

The related notion of thin bases is the following

Definition: A' is said to be *thin complete subsequence* of A if A' is complete and $A'(x) = (1 + o(1)) \log_2 x$.

We shall show not only that the sequence of squares S has a thin complete subsequence but a wild class of complete sequence has this property. We prove

Theorem: Let $A = \{a_1 < a_2 < \ldots\}$ be a complete sequence of integers. Assume that $\lim_{n \to \infty} a_{n+1}/a_n = 1$. Then A contains a thin complete subsequence.

The proof will be completely constructive. Let $X = \{x_1 < x_2 < \ldots\}$. Let us denote

$$G(X) = \sup_i \{x_{i+1} - x_i\}.$$

(So that if $G(X) < \infty$ then it indicates the size of the biggest gap in X)

Proof of the Theorem:

We need some lemmas.

Lemma 1: Let $X = \{x_1 < \ldots < x_n < \ldots\} \in \mathbf{N}$. Assume that for every $i = 0, 1, \ldots$

$$x_{i+1} \leq x_1 + \ldots + x_i. \tag{1}$$

Then $G(\Sigma(X)) \leq x_1$.

The proof of Lemma 1 is easy or see [3].

Lemma 2: Let A be a complete sequence of integers. Let $A_3 = \{2\Delta_A < a'_1 < a'_2 < \ldots\}$ be an infinite sequence of A for which $a'_{i+1}/a'_i \le 2$ $i = 1, 2, \ldots$. Assume there are finite subsequence A_1, A_2 of A for which $\Delta_A \le S(A_2) - s(A_1) < a'_1 - \Delta_A$. Finally let $A_4 = A \cap [1, a'_1)$ and let us assume the sets $A_1,\ A_2,\ A_3,\ A_4$ pairwise disjoint. Then $B = \bigcup_{i=1}^{4} A_i$ is complete.

Proof of Lemma 2: Let $b_i = S(A_i)$, $i = 1, 2$ and $\Sigma(A_1) = \{a'_1 = x_1 < x_2 < \ldots\}$. We shall prove the existence of Δ_B which clearly proves Lemma 2. We prove

$$\Delta_B \le \Delta_A + (a'_1 + b_2). \tag{2}$$

(Thus we have to show if $n \ge \Delta_A + (a'_1 + b_2)$ then $n \in \Sigma(B)$)

First we show

$$G(\Sigma(A_1)) \le a'_1. \tag{3}$$

By Lemma 1 we need that for every $i = 0, 1, \ldots$

$$a'_{i+1} \le a'_1 + a'_2 + \ldots + a'_i. \tag{4}$$

For $i = 0$ (4) is trivial and since $a'_{i+2} \le 2a'_{i+1}$ we get

$$a'_{i+2} \le 2a_{i+1} \le (a'_1 + \ldots + a'_i) + a'_{i+1}$$

providing (4) holds for i. Thus by induction we infer that (3) holds.

Now let

$$m_i = \min_j \{0 \le n - (x_j + b_i) : x_j \in \Sigma(A_1)\}$$

$i = 1, 2$

First let us observe

$$0 \le m_i < a'_1. \tag{5}$$

Indeed

$$G((\Sigma(A_1) + b_i \cap [b_i, \infty)) = G(\Sigma(A_1)) \le a'_1$$

for $i = 1, 2$. Furthermore $0 \le m_i$ follows from the definition of m_i. If $m_2 \ge \Delta_A$ then by (5) we get $m_2 \in [\Delta_A, \Delta_A + 1, \ldots, a'_1) \subset \Sigma(A_4)$, which means $n - (x_j + b_2) \in \Sigma(A_4)$ or $n \in \Sigma(A_2) + \Sigma(A_3) + \Sigma(A_4) \subseteq \Sigma(B)$ since A_2, A_3 and A_4 pairwise disjoint.

So let us assume $0 \le m_2 < \Delta_A$. Then

$$\Delta_A \le b_2 - b_1 \le m_2 + (b_2 - b_1) < \Delta_A + a'_1 - \Delta_A = a'_1$$

and so

$$\Delta_A \le m_2 + (b_2 - b_1) = n - (x_j + b_2) + b_2 - b_1 = n - (x_j + b_1) < a'_1$$

thus we have

$$n - (x_j + b_1) \in \Sigma(A_4)$$

and so

$$n \in \Sigma(A_4) + (x_j + b_1) \subset \Sigma(A_1) + \Sigma(A_3) + \Sigma(A_4) \subseteq \Sigma(B).$$

This completes the proof of Lemma 2.

Now we turn to the proof of the theorem.

Let $A = \{a_1 < a_2 < \ldots\}$ with

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1. \tag{6}$$

and let Δ_A be the threshold of completeness of A. We are going to construct a thin complete subsequence B of A. B will be composed by four pairwise disjoint subsequence of A For the brevity let $\Delta = \Delta_A$ Clearly $a_{n+\Delta} - a_n \geq \Delta$ and by (6)

$$\lim_{n\to\infty} \frac{a_{n+\Delta}}{a_n} = \lim_{n\to\infty} \frac{a_{n+\Delta}}{a_{n+\Delta-1}} \cdots \frac{a_{n+1}}{a_n} = 1.$$

This implies there is a m for which

$$2\Delta < a_{m-1} < a_m < a_{m+\Delta}$$

and $a_{m+\Delta} < 2\max\{a_t : a_t \leq 2a_{m-1}\} - \Delta$. Let $a'_1 = a_{m-1}$; $A_2 = \{a_{m+\Delta}\}$, $A_1 = \{a_m\}$ and $A_4 = A \cap [1, a'_1)$.

Assume $a'_1 \in A_3$ and and the elements $a'_1 < \ldots < a'_k$ have been defined. Then let

$$a'_{k+1} = \max\{a_t : a_t \leq 2a'_k\}. \tag{7}$$

Finally let $A_3 = \{a'_1 < a'_2 < \ldots\}$.

Clearly the sets A_1, A_2, A_3 and A_4 pairwise disjoint and the conditions of Lemma 2 hold for these sets. Cosequently $B = \bigcup_{i=1}^4 A_i$ is complete.

In the rest of the proof we shall show that B is thin.

Let $K = |A_1| + |A_2| + |A_4|$. ($A_1, A_2, A_4$ are finite sets so K is a natural number). Thus

$$B(x) \leq K + A_3(X),$$

which means that if A_3 is thin so is B.

Now we estimate $A_3(x)$.

We have $\lim_{n\to\infty} \frac{a_{n+\Delta_A}}{a_n} = 1$ consequently by (7) $\lim_{n\to\infty} a'_{n+1}/a'_n = 2$ or $\lim_{m\to\infty} \log_2 a'_m/m = 1$.

Let $a'_m \leq x < a'_{m+1}$. Then

$$\frac{\log_2 a'_m}{m} \leq \frac{\log_2 x}{m} = \frac{\log_2 x}{A_3(x)} < \frac{\log_2 a'_{m+1}}{m}$$

thus $\lim_{x\to\infty} A_3(x)/\log_2 x = 1$ so that A_3 is thin.

2. Concluding remarks

Applying the theorem we have there are many "classical" sequences which have thin complete subsequences. Let

$$P = \{2 < 3 < \ldots < p_n < \ldots\}$$

be the sequence of prime numbers, let

$$B(p,q) = \{p^k q^m : (p,q) = 1;, \ p,q > 1, \ p,q,m,k \in \mathbf{N}\}$$

be the sequence the Birch-sequence.

Let $g_m(x) \in \mathbf{Z}[x]$. Assume $g_m(x)$ has positive leading coeffitient and $g.c.d.\{g_m(1), \ldots, g_m(n), \ldots\} = 1$ and let

$$G = \{g_m(1), \ldots, g_m(n), \ldots\}. \tag{8}$$

Richert proved in [8] that $\Delta_P = 6$ and it is well known that $p_{n+1}/p_n \to 1$ as $n \to \infty$.

Erdős conjectured and Birch proved that $B(p,q)$ is complete sequence (see [5]). By the irrationality of $(\log p/\log q)$ we infer that the quotient of consecutive terms of this sequence tends to 1 .

Finally the completeness of sequences type (8) were investigated by many authors. For instance in 1948 Sprague proved for the sequence of squares that $\Delta_S = 128$ [6]. Further he proved in [7] that for every k the sequence $\{n^k : n \in N\}$ is complete. A far-reaching generalization of Birch's and Sprague's results was published by J. W. Cassels (see [4] and (5)). This result gives in the general case that the sequence G is complete.

Furthermore since

$$\lim_{n\to\infty} g_m(n+1)/g_m(n) = 1$$

we conclude that for these sequences we could use the Theorem and we have

Corollary: The sequences $P, B(p,q)$ and G have thin complete subsequence.

Certainly there exists complete sequence which has no thin complete subsequence. For instance if $\Phi = \{F_1 < \ldots\}$ where $F_1 = 1, F_2 = 2$ is the sequence of Fibonacci then it is well known that Φ is complete and $\Phi(x) = c\log_2 x; . \ c > 1$. But if we omit at least two elements from Φ then the remaining sequence cannot be complete (see [5]).

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