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On monochromatic sums of squares and primes

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Abstract

Let K be a positive integer. A partition $\{\mathcal{A}_k,\ 1 \leqslant k \leqslant K\}$ of the sequence of squares being given, we consider the question of estimating the smallest number $t(K)$ such that any large integer can be written as a sum of less than $t(K)$ elements all taken from one of the sets $\mathcal{A}_k$. The analogous question for the primes is also tackled.

1. Introduction

A set $\mathcal{A}$ of positive integers is called an asymptotic basis of order h if any large enough integer is a sum of at most h elements of $\mathcal{A}$, the integer h being the least one such that this property holds. In [5], A. Sárközy considered the problem of estimating the maximal order $H(k)$, as asymptotic bases, of the subsequences of primes having a positive relative density $1/k$. He obtained the upper bound $H(k) \ll k^4$ and the lower bound $H(k) \gg k \log\log k$. Later Ramaré and Ruzsa improved almost definitively this result by showing $H(k) \asymp k \log\log k$ (cf. [3]). In fact they obtained a much more general result which applies to a large family of asymptotic bases, but unfortunately not to the sequence of squares.

A related question concerns the representation of the integers as monochromatic sums of elements of a given basis: let $\mathcal{A}$ be an asymptotic basis. In [6, p. 26], A. Sárközy wrote the

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following lines: "*It is easy to see that for all $k \in \mathbb{N}$, there is a number $t = t(k)$ with the property that for any k-colouring of the set of squares every integer large enough can be represented as the monochromatic sum of at most t squares*." He then proposed the problem of estimating the smallest number $t = t(k)$ having this property, and also the similar problem for the primes.

A partition of the primes into K sets being given, it is not true one of the sets is uniformly dense in the set of the primes. For that reason we do not see how Ramaré–Ruzsa's result could be applied to this problem.

In this note, we consider these problems, obtaining in both cases nontrivial upper and lower bounds by elementary means. For this, we will follow more or less Sárközy's approach in [5] except that, instead of Kneser theorem, we will employ a finite addition theorem due to Sárközy himself (cf. [4]).

In a last section, we study a more general question, stating a theorem which applies to a large family of bases, especially to any classical bases including the kth powers.

Let s be a positive integer. For any integer n, we denote by $r_{\mathcal{A}}^{(s)}(n)$ the number of s-tuples $(a_1, a_2, \dots, a_s)$ of elements of $\mathcal{A}$ such that

$$n = a_1 + a_2 + \cdots + a_s.$$

The s-fold sumset $s\mathcal{A}$ is the set of all integers n such that $r_{\mathcal{A}}^{(s)}(n) > 0$. A positive integer k being given, the set of its positive multiples will be denoted by $\mathbb{N}k$.

For any integer positive K and any K-partition $\mathcal{U} = (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_K)$ of $\mathcal{A}$ as a union of K sets

$$\mathcal{A} = \bigcup_{k=1}^{K} \mathcal{A}_k,$$

we denote by $\operatorname{ord}(\mathcal{U})$ the least number h having the following property: for any sufficiently large integer n there exists k such that n is a sum of at most h elements of $\mathcal{A}_k$. We finally denote

$$\operatorname{ord}_K(\mathcal{A}) = \sup\bigl\{\operatorname{ord}(\mathcal{U})\colon \mathcal{U} \text{ is a } K\text{-partition of } \mathcal{A}\bigr\}.$$

The key tool is the following result of Sárközy which can be viewed as a finite Kneser type theorem.

Lemma 1. *Let N and k be positive integers and $\mathcal{A} \subset \{1, 2, \dots, N\}$ such that*

$$|\mathcal{A}| > \frac{N}{k} + 1.$$

Then there exist integers d, h, m such that

$$1 \leqslant d \leqslant k - 1,$$
$$1 \leqslant h \leqslant 118k,$$

and

$$\bigl\{(m+1)d, (m+2)d, \dots, (m+N)d\bigr\} \subset h\mathcal{A}.$$

2. The squares

We denote by $\mathcal{Q}$ the set of squares. We will use the following bound

Lemma 2. *For any $n \geqslant 1$, we have*

$$r_{\mathcal{Q}}^{(5)}(n) \leqslant 30n^{3/2}. \tag{1}$$

Proof. By [1, Theorem 4, p. 180], we have

$$r_{\mathcal{Q}}^{(5)}(n) = \frac{4\pi^2}{3}\mathfrak{S}(n)n^{3/2},$$

where

$$\mathfrak{S}(n) = \sum_{q \geqslant 1} T_n(q) = \prod_p \sum_{h \geqslant 0} T_n\big(p^h\big),$$

with

$$T_n(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} T(a,q)^5 e^{-i2\pi an/q}, \quad T(a,q) = \frac{1}{q}\sum_{x=1}^{q} e^{i2\pi ax^2/q}.$$

Using the estimates of $T(a,q)$ given in [1, Theorem 3, p. 138], we easily show that uniformly $\mathfrak{S}(n) \leqslant 2.27$, yielding the upper bound (1). □

Let

$$\mathcal{Q} = \bigcup_{k=1}^{K} \mathcal{Q}_k, \tag{2}$$

be a partition of the squares. Let N_0 be an integer large enough such that for any $N \geqslant N_0$

$$\pi(\sqrt{N}) - \pi(\sqrt{N}/2) \geqslant K + 1.$$

Take any $N \geqslant N_0$ and put

$$\mathcal{S}_k = \mathcal{Q}_k \cap [N/4, N], \quad k = 1, 2, \ldots, K.$$

For each prime p, let

$$I_p = \{1 \leqslant k \leqslant K \colon \mathcal{S}_k \subset \mathbb{N}p\}.$$

We then define recursively the following, possibly empty, increasing sequence of prime numbers:

$$q_1 = \min\{p \colon I_p \neq \emptyset\},$$
$$q_j = \min\big\{p \colon I_p \setminus (I_{q_1} \cup \cdots \cup I_{q_{j-1}}) \neq \emptyset\big\}, \quad j \geqslant 2.$$

This sequence is clearly finite: $q_1 < q_2 < \cdots < q_r$ with $|I_{q_1} \cup \cdots \cup I_{q_r}| \leqslant K - 1$. We denote $\mathcal{K}'$ the complementary set of $I_{q_1} \cup \cdots \cup I_{q_r}$ in $\{1, 2, \ldots, K\}$. We have

$$\begin{aligned}\Bigl|\bigcup_{k\in\mathcal{K}'} \mathcal{S}_k\Bigr| &\geqslant \prod_{j=1}^{r}\Bigl(1-\frac{1}{q_j}\Bigr)\frac{\sqrt{N}}{2}\\ &\geqslant \prod_{j=1}^{K-1}\Bigl(1-\frac{1}{p_j}\Bigr)\frac{\sqrt{N}}{2}\\ &\geqslant \frac{\sqrt{N}}{4\log K},\end{aligned}$$

by an explicit lower bound in Mertens' formula, where $p_1 < p_2 < \cdots$ is the increasing sequence of prime numbers.

Hence there exists some $k \in \mathcal{K}'$ such that

$$|\mathcal{S}_k| \geqslant \frac{\sqrt{N}}{4K\log K}.$$

Put $\mathcal{S} = \mathcal{S}_k$. We have

$$\Bigl(\sum_{s\in\mathcal{S}} 1\Bigr)^5 = |\mathcal{S}|^5 \geqslant \Bigl(\frac{\sqrt{N}}{4K\log K}\Bigr)^5,$$

and on the other hand,

$$\begin{aligned}\Bigl(\sum_{s\in\mathcal{S}} 1\Bigr)^5 &= \sum_{n\in 5\mathcal{S}} r_{\mathcal{S}}^{(5)}(n) \leqslant \sum_{n\in 5\mathcal{S}} r_{\mathcal{Q}}^{(5)}(n)\\ &\leqslant |5\mathcal{S}| \max_{1\leqslant n\leqslant 5N} r_{\mathcal{Q}}^{(5)}(n) \leqslant 340N^{3/2}|5\mathcal{S}|,\end{aligned}$$

by (1), where we write $5\mathcal{S}$ for denoting the set of the sums of 5 elements from $\mathcal{S}$. It satisfies $5\mathcal{S} \subset [5N/4, 5N]$ and

$$|5\mathcal{S}| \geqslant \frac{N}{c_1(K\log K)^5},$$

for some absolute constant $c_1 > 0$. Assuming N large enough, we deduce from Lemma 1 that there exist d with $1 \leqslant d \leqslant c_1(K\log K)^5$ such that for some

$$h \leqslant h_0 = c_2(K\log K)^5,$$

we have

$$\bigl\{(m+1)d, (m+2)d, \ldots, (m+5N)d\bigr\} \subset 5h\mathcal{S},$$

for some m such that

$$\frac{5hN}{4} \leqslant md \quad \text{and} \quad (m+5N)d \leqslant 5hN.$$

Since k belongs to $\mathcal{K}'$, we see that $\mathcal{S} = \mathcal{S}_k$ contains some integer s coprime to d and satisfying

$$\frac{N}{4} \leqslant s \leqslant N.$$

Thus any integer in the interval

$$\mathcal{L} := \{(m+1)d + (d-1)N, (m+2)d + (d-1)N + 1, \dots, (m+5N)d\}$$

can be written as a sum $x + js$ where $x \in 5h\mathcal{S}$ and $0 \leqslant j \leqslant d-1$. By shifting $\mathcal{L}$ by multiples of s and taking the union of the given intervals $\mathcal{L} + js$, $0 \leqslant j \leqslant l$, we get

$$\big[(m+N)d, (m+5N)d + lN/4\big] \subset \bigcup_{j=5h}^{5h+d-1+l} j\mathcal{S}.$$

Applying this argument to $N+1$ instead of N, we get for any $l' \geqslant 0$

$$\big[(m'+N+1)d', (m'+5N+5)d' + l'(N+1)/4\big] \subset \bigcup_{j=5h'}^{5h'+d'-1+l'} j\mathcal{S}',$$

where

$$\mathcal{S}' = \mathcal{Q}_{k'} \cap \left(\frac{N+1}{4}, N+1\right], \quad 1 \leqslant k' \leqslant K, \quad 1 \leqslant d' \leqslant c_1(K\log K)^5, \quad 1 \leqslant h' \leqslant h_0,$$

and

$$(m'+N+1)d' \leqslant (5h'-4d')(N+1) \leqslant (5h'-4)(N+1) \leqslant (5h_0-4)(N+1).$$

Since $(m+5N)d + lN/4 \geqslant 5Nd + lN/4$, letting $l = l(N) = 20h_0 - 20d - 15$, it follows that the intervals $I(N) = [(m+N)d, (m+5N)d + lN/4]$, N sufficiently large, where m, d depend on N, overlap. Thus any large integer is a monochromatic sum in terms of partition (2) of at most $25h_0 = c_3(K\log K)^5$ squares. We thus have proved:

Theorem 1. *Let K be an integer. Then*

$$\operatorname{ord}_K(\mathcal{Q}) \leqslant c_3(K\log K)^5,$$

where c_3 can be taken equal to 10^9.

We now turn to obtain a lower bound of $\mathrm{ord}_K(\mathcal{Q})$.

For any $s \geqslant 2$, let $M_s = p_1 p_2 p_3 \cdots p_s$ where $p_1 < p_2 < p_3 < \cdots$ is the increasing sequence of prime numbers. We denote by R the set of all non-zero quadratic residues modulo M_s. Then

$$|R| = \frac{p_1 - 1}{2} \cdot \frac{p_2 - 1}{2} \cdot \cdots \cdot \frac{p_s - 1}{2}.$$

Let us consider the following partition of the squares:

$$\mathcal{Q} = \bigcup_{j=1}^{s} \{m^2\colon\ (m, M_{j-1}) = 1 \text{ and } p_j \mid m\} \cup \bigcup_{m \in R} \mathcal{Q} \cap (m + \mathbb{N}M_s).$$

This a K_s-partition with

$$K_s = s + \frac{p_1 - 1}{2} \cdot \frac{p_2 - 1}{2} \cdot \cdots \cdot \frac{p_s - 1}{2}.$$

Let n be a large square free multiple of M_s. If h is such that $h(m + qM_s) = n$ for some $m \in R$, then $M_s \mid h$. This yields $h \geqslant M_s$. We obtain

$$\mathrm{ord}_{K_s}(\mathcal{Q}) \geqslant M_s.$$

Now let $K \geqslant 2$ be an integer. Then there is an $s \geqslant 1$ such that $K_s \leqslant K < K_{s+1}$. Since $(\mathrm{ord}_K(\mathcal{Q}))_{K \geqslant 1}$ is not decreasing, we get

$$\mathrm{ord}_K(\mathcal{Q}) \geqslant \mathrm{ord}_{K_s}(\mathcal{Q}) \geqslant M_s = \frac{M_{s+1}}{p_{s+1}} \geqslant \frac{2^{s+1} K_{s+1}}{p_{s+1}} > \frac{2^{s+1} K}{p_{s+1}}.$$

Classic asymptotic estimates on the primes give

$$s + 1 = \bigl(1 + o(1)\bigr) \frac{\log K}{\log\log K} \quad \text{and} \quad p_{s+1} = e^{(1+o(1)) \log s} = e^{(1+o(1)) \log\log K},$$

thus we finally deduce the following lower bound:

Theorem 2. *Let K be an integer. Then*

$$\mathrm{ord}_K(\mathcal{Q}) \geqslant K \exp\left(\bigl(\log 2 + o(1)\bigr) \frac{\log K}{\log\log K}\right).$$

3. The primes

We denote by $\mathcal{P}$ the set of all prime numbers. The starting point could be the following upper bound for the number of representations of an integer as a sum of 3 primes.

Lemma 3. (Cf. [2].) *Let N be a large integer. Then for any $n \leqslant N$, we have*

$$r_{\mathcal{P}}^{(3)}(n) \ll \frac{N^2}{(\log N)^3}.$$

An alternative way is to use the following average property, yielding a better constant in the final upper bound for $\mathrm{ord}_K(\mathcal{P})$. For any set of integers $\mathcal{S}$, we denote by $\rho^{(2)}(\mathcal{S})$ the number of solutions $s_1, s_2, s_3, s_4 \in \mathcal{S}$ of the equation $s_1 + s_2 = s_3 + s_4$.

Lemma 4. (Cf. [2].) *Let N be a large integer. Then*

$$\rho^{(2)}\big(\mathcal{P} \cap (N/2, N]\big) \leqslant \frac{N^3}{5(\log N)^4}. \tag{3}$$

Proof. Using some classic upper bound for $r_{\mathcal{P}}^{(2)}(n)$ deduced from a sifting approach, we may obtain such an upper bound for $\rho^{(2)}(\mathcal{P} \cap (N/2, N])$ which differs from the stated result by a constant (see for example [2, Lemma 7.7]). Another way to proceed consists in using the circle method for estimating this number $\rho^{(2)}(\mathcal{P} \cap (N/2, N])$. On the major arcs, we will obtain the contribution

$$\big(1 + o(1)\big)\frac{\mathfrak{S}\mathcal{I}(N)}{(\log N)^4},$$

where

$$\mathfrak{S} = \prod_p \left(1 + \frac{1}{(p-1)^3}\right) < \frac{12}{5},$$

and

$$\mathcal{I}(N) = \sum_{\substack{N/2 < a,b,c,d \leqslant N \\ a+b=c+d}} 1 = \frac{N^3}{12} + O\big(N^2\big).$$

On the minor arcs, the contribution is negligible. It follows that (3) holds for any sufficiently large integer N. □

Let

$$\mathcal{P} = \bigcup_{k=1}^{K} \mathcal{P}_k, \tag{4}$$

be a partition of the primes. By the prime number theorem, since $20^{1/4} > 2$, we can find an integer N_0 such that for any $N \geqslant N_0$, both (3) and

$$\pi(N) - \pi(N/2) \geqslant \frac{N}{20^{1/4}\log N} \tag{5}$$

are satisfied. Let $N \geqslant N_0$ and put

$$\mathcal{S}_k = \mathcal{P}_k \cap (N/2, N], \quad k = 1, 2, \ldots, K.$$

For any $k = 1, \dots, K$, we have by Cauchy–Schwarz inequality,

$$|\mathcal{S}_k|^4 \leqslant |2\mathcal{S}_k|\rho^{(2)}(\mathcal{S}_k),$$

thus there exists k such that

$$|2\mathcal{S}_k| \geqslant \frac{|\mathcal{S}_1|^4 + \cdots + |\mathcal{S}_K|^4}{\rho^{(2)}(\mathcal{S}_1) + \cdots + \rho^{(2)}(\mathcal{S}_K)} \geqslant \frac{|\mathcal{S}_1|^4 + \cdots + |\mathcal{S}_K|^4}{\rho^{(2)}(\mathcal{P} \cap (N/2, N])}.$$

By Hölder inequality we get

$$|2\mathcal{S}_k| \geqslant \frac{(|\mathcal{S}_1| + \cdots + |\mathcal{S}_K|)^4}{K^3\rho^{(2)}(\mathcal{P} \cap (N/2, N])} = \frac{(\pi(N) - \pi(N/2))^4}{K^3\rho^{(2)}(\mathcal{P} \cap (N/2, N])}$$

giving by Lemma 4 and (5)

$$|2\mathcal{S}_k| \geqslant \frac{N}{4K^3}.$$

We put $\mathcal{S} = \mathcal{S}_k$. Since $2\mathcal{S} \subset (N, 2N]$, applying Lemma 1 to $2\mathcal{S} - N$ shows for N large enough that there exists an integer d with $1 \leqslant d \leqslant 4K^3$ such that for some

$$h \leqslant h_0 = 500K^3, \tag{6}$$

we have

$$hN + \{(m+1)d, (m+2)d, \dots, (m+2N)d\} \subset 2h\mathcal{S},$$

for some m such that $(m + 2N)d \leqslant hN$. Since $\mathcal{S}$ contains at least two primes, we can find a prime p in $\mathcal{S}$ which is coprime to d. Thus the following interval of consecutive integers

$$hN + \{(m+1)d + (d-1)N, (m+2)d + (d-1)N + 1, \dots, (m+2N)d\}$$

is contained in $\bigcup_{j=2h}^{2h+d-1} j\mathcal{S}$. Now shifting this interval by successive multiples of some arbitrary element $p \in \mathcal{S}$, we get

$$hN + [(m+N)d, (m+2N)d + lN] \subset \bigcup_{j=2h}^{2h+d-1+2l} j\mathcal{S}.$$

Applying this with $N + 1$ instead of N, we get for any $l' \geqslant 0$,

$$h'(N+1) + [(m'+N+1)d', (m'+2N+2)d' + l'(N+1)] \subset \bigcup_{j=2h'}^{2h'+d-1+2l'} j\mathcal{S}',$$

where

$$\mathcal{S}' = \mathcal{P}_{k'} \cap \big((N+1)/2, N+1\big], \quad 1 \leqslant k' \leqslant K, \quad 1 \leqslant d' \leqslant 4K^3, \quad 1 \leqslant h' \leqslant h_0,$$

and

$$h'(N+1) + (m' + N + 1)d' \leqslant (2h' - d')(N+1) \leqslant (2h_0 - 1)(N+1).$$

Since $hN + (m+2N)d + lN \geqslant (h+l+2d)N$, we get for $l = 2h_0 - 2d - h$

$$h'(N+1) + (m' + N + 1)d' \leqslant hN + (m+2N)d + lN.$$

It follows that we can cover all sufficiently large integers by sums of at most $3h_0$ monochromatic sums of primes, according to the given partition (4).

In view of (6), we thus have proved the following result:

Theorem 3. *Let K be an integer. Then*

$$\mathrm{ord}_K(\mathcal{P}) \leqslant 1500K^3.$$

We now show a nontrivial lower bound of $\mathrm{ord}_K(\mathcal{P})$. For any integer $M \geqslant 1$, we consider the partition

$$\mathcal{P} = \{p \in \mathcal{P}\colon\ p \mid M\} \cup \bigcup_{\substack{m=1\\(m,M)=1}}^{M} \mathcal{P} \cap (m + \mathbb{N}M)$$

and the colouring classes induced by it. This is a K-partition with

$$K = 1 + \varphi(M),$$

where φ is the Euler's totient function. Let us count the minimal number of monochromatic summands needed to represent a large positive integer n congruent to 0 modulo M: it is clearly sufficient to consider the chromatic classes $\mathcal{P} \cap (m + \mathbb{N}M)$, where $(m, M) = 1$. Obviously any integer h such that $h(m + qM) = n$ for some m coprime to M and some $q \geqslant 0$ must satisfy $M \mid h$. Thus

$$\mathrm{ord}_{1+\varphi(M)}(\mathcal{P}) \geqslant M. \tag{7}$$

Now let $K \geqslant 2$ be any integer. Let the sequence $(M_s)_{s\geqslant 1}$ be defined as in the previous section. There exists an $s \geqslant 1$ such that $1 + \varphi(M_s) \leqslant K < 1 + \varphi(M_{s+1})$, or equivalently

$$p_s - 1 \leqslant \frac{K-1}{\varphi(M_{s-1})} < (p_s - 1)(p_{s+1} - 1).$$

Let λ be the integral part of $\frac{K-1}{\varphi(M_{s-1})}$. Observe that $\lambda \geqslant p_s - 1$. We thus have

$$(\lambda + 1)\varphi(M_{s-1}) > K - 1 \geqslant \lambda\varphi(M_{s-1}) \geqslant \varphi(\lambda M_{s-1}).$$

Since the sequence $(\mathrm{ord}_K(\mathcal{P}))_{K\geqslant 1}$ is nondecreasing, we deduce from (7)

$$\begin{aligned}\mathrm{ord}_K(\mathcal{P}) \geqslant \mathrm{ord}_{1+\varphi(\lambda M_{s-1})}(\mathcal{P}) \geqslant \lambda M_{s-1} &= \left(\frac{\lambda}{\lambda+1}\right)\frac{(\lambda+1)\varphi(M_{s-1})}{\prod_{p|M_{s-1}}(1-\frac{1}{p})}\\ &> \left(\frac{p_s-1}{p_s}\right)\frac{K-1}{\prod_{p|M_{s-1}}(1-\frac{1}{p})}.\end{aligned}$$

From Mertens' formula, we obtain

$$\prod_{p|M_{s-1}}\left(1-\frac{1}{p}\right) = \frac{e^{-\gamma}+o(1)}{\log s} = \frac{e^{-\gamma}+o(1)}{\log\log K},$$

by using the estimate

$$\log K = \bigl(1+o(1)\bigr)\log\varphi(M_s) = \bigl(1+o(1)\bigr)\log M_s = \bigl(1+o(1)\bigr)p_s = \bigl(1+o(1)\bigr)s\log s,$$

deduced from the prime number theorem. We thus have the following lower bound:

Theorem 4. *Let K be an integer. Then*

$$\mathrm{ord}_K(\mathcal{P}) \geqslant \bigl(e^{\gamma}+o(1)\bigr)K\log\log K.$$

Note that Sárközy in [5, Theorem 10] used a similar approach to get a lower bound for the order as additive basis of a dense set of primes.

4. Sufficiently well sifted bases

It is clear by considering the following basis

$$\mathcal{A} = \{1\}\cup\{hq\colon\ q\geqslant 0\},$$

that not every basis has a K-chromatic order. The main reason is that the elements of $\mathcal{A}$ are not sufficiently well sifted in the sense explained below. In order to avoid such degenerate cases, we need to introduce a certain class of bases.

Let $\mathcal{A}$ be a basis. For any prime p, we denote by $\mathcal{A}^{(p)} = \{a\in\mathcal{A}\colon\ p\mid a\}$, and we put

$$\mathcal{A}^{(q_1,q_2,\ldots,q_K)} = \bigcup_{j=1}^{K}\mathcal{A}^{(q_j)},$$

for any set $q_1,q_2,\ldots,q_K$ of primes.

Let $\tau<1$. We say that $\mathcal{A}$ is well τ-sifted if there exists a real number $\tau=\tau(\mathcal{A},K)>0$ depending only on $\mathcal{A}$ and K (and not on the sequence $q_1,\ldots,q_K$ of primes) such that

$$\limsup_{N\to\infty}\frac{A^{(q_1,q_2,\ldots,q_K)}(N)}{A(N)} \leqslant 1-\tau.$$

It is not hard to adapt the proofs of Theorems 1 and 3 in order to obtain:

Theorem 5. *Let $\mathcal{A}$ be a basis which is well τ-sifted for some $\tau = \tau(\mathcal{A}, K) > 0$. Assume that there exist a positive integer s and a constant $C = C(\mathcal{A}, s)$ such that*

(H1) *either* $r_{\mathcal{A}}^{(s)}(n) \leqslant \frac{CA(N)^s}{N}$, *for any* $n \leqslant sN$ *and any sufficiently large* N,
(H2) *or* $\sum_{n=1}^{N} (r_{\mathcal{A}}^{(s)}(n))^2 \leqslant \frac{CA(N)^{2s}}{N}$, *for any sufficiently large* N.

Then $\mathrm{ord}_{\mathrm{K}}(\mathcal{A})$ *is finite. More precisely*

$$\mathrm{ord}_{\mathrm{K}}(\mathcal{A}) \leqslant \frac{C' K^s}{\tau(\mathcal{A}, K)^s} \quad \textit{under } (\mathrm{H1}), \quad \textit{and} \quad \mathrm{ord}_{\mathrm{K}}(\mathcal{A}) \leqslant \frac{C'' K^{2s-1}}{\tau(\mathcal{A}, K)^{2s}} \quad \textit{under } (\mathrm{H2}).$$

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