ADDITIVE STRUCTURE OF DIFFERENCE SETS AND A THEOREM OF FØLNER

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Abstract. A theorem of Følner asserts that for any set $A \subset \mathbb{Z}$ of positive upper density there is a Bohr neigbourhood B of 0 such that $B \setminus (A - A)$ has zero density. We use this result to deduce some consequences about the structure of difference sets of sets of integers having a positive upper density.

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1. Introduction

This paper is about the structure of the difference set $D(A) := A - A$ of sets of integers having positive density. By density we mean the upper asymptodic density defined by

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap [-n, n]|}{2n + 1} > 0.$$

For sets $X, Y \subseteq \mathbb{Z}$ we mean

$$X + Y = \{x + y : x \in X; y \in Y\}.$$

and

$$X \cdot Y = \{x \cdot y : x \in X; y \in Y\}.$$

We define a *Bohr set* as a set of the form

$$B(S, \varepsilon) = \{m \in \mathbb{Z} : \max_{s\in S} \|sm\| < \varepsilon\}, \tag{1.1}$$

where S is a finite set of real numbers. Here $\|x\| = \min_{n\in\mathbb{Z}} |x - n|$, the absolute fractional part.

Recall that for every Bohr set has positive density, and for every pair of sets S, S' and for every k, $0 < k \cdot \varepsilon' \le \varepsilon$, we have

$$k \cdot B(S, \varepsilon') \subseteq B(S, \varepsilon), \tag{1.2}$$

and

$$B(S \cup S', \varepsilon) = B(S, \varepsilon) \cap B(S', \varepsilon) \tag{1.3}$$

(see e.g. [8] p. 165).

These sets are just the basic neighbourhoods of 0 in the Bohr topology. We say this is a k, ε-neighbourhood if $|S| = k$ (or a k-neighbourhood if ε is unimportant).

Bogolyubov [3] proved for the case of integers, and Følner [4], [5] generalized for general commutative groups, that the second difference set $A - A + A - A$ of a set having positive upper Banach density is always a Bohr neighborhood of 0.

In Bogolyubov's theorem four copies of A are used. Three suffice with a small change. If r, s, t are nonzero integers satisfying $r+s+t = 0$ and A is a set of integers having positive Banach density, then $S = rA + sA + tA$ is a Bohr neighbourhood of 0, see [2]. Here $rA = \{rb : b \in A\}$. $r+s+t = 0$ is necessary to exclude trivial counterexamples; so there is no really "symmetric" result here. (A further comment on this is given in Section 3). The case $r = s = 1$, $t = -2$ immediately generalizes Bogolyubov's theorem.

On the other hand, a theorem of Kříž([7]) implies that there is a set A with positive upper density whose difference set contains no Bohr set.

In the positive direction in [4] Følner proved that there is a Bohr set which is almost a subset of $A - A$; the exceptional set has zero density.

In this paper we give some applications of the Følner theorem. We are going to investigate $A + A + A$ and $A + A - A$ and Bohr sets. In [1] V. Bergelson investigated the additive structure of $D(A)$. He also proved that there exists an infinite set B of integers for which $A - A \supseteq B + B + \cdots + B = k \cdot B$, provided A has positive upper density. His proof of this theorem is based on an ergodic theorem. In [6] the first author gave a purely combinatorial proof for this result. Here we give a third proof for it using Følner's theorem.

2. Applications of Følner's theorem

We have already mentioned in the introduction that $D(D(A))$ always contains a Bohr set, while the set $D(A)$ does not necessary contain a Bohr set. Now we investigate the three-fold sum-differences of A.

Theorem 2.1. *There is a symmetric set A of integers such that $0 \in A$, $\overline{d}(A) > 0$ and the set $A + A + A$ does not contain a Bohr set.*

On the other hand we prove that $A + A - A$ is always a Bohr neighborhood of many $a \in A$.

Theorem 2.2. *Assume that $\overline{d}(A) > 0$. There exists a subset A' of A, such that $d(A \setminus A') = 0$ and for every $a' \in A'$, the set $A + A - A - a'$ is a Bohr neighbourhood of 0.*

This generalizes Bogolyubov's theorem in a different direction.

Note that in the same way one can prove Theorem 2.2 in the following form: For every A and X, $\overline{d}(A) > 0, \overline{d}(X) > 0$ there exists a subset

X' of X, such that $d(X \setminus X') = 0$ and for every $x' \in X'$, the set $X + A - A - x'$ is a Bohr neighbourhood of 0.

Proof of Theorem 2.1. By the theorem of Kříž we know the existence of a set X of positive integers for which $\overline{d}(X) > 0$, and the set $X - X$ does not contain a Bohr set. Let

$$Y = \{4x + 1 : x \in X\},$$

and

$$A = Y \cup -Y \cup \{0\}.$$

Since $\overline{d}(Y) = \frac{1}{4}\overline{d}(X) > 0$, we have $\overline{d}(A) > 0$ and the set A is symmetric and contains 0.

Now we prove that $A + A + A$ does not contain a Bohr set. Assume to the contrary there is a $B(S, \varrho) \subseteq A + A + A$. Since $4B(S, \varrho/4) \subseteq B(S, \varrho) \subseteq A + A + A$ also holds we get there is a Bohr set $B(S', \varepsilon)$ in $A + A + A \cap \{4k : k \in \mathbb{Z}\}$. $4k \in A + A + A$ holds if and only if $4k \in Y - Y$. But

$$Y - Y = \{4x - 4x' : x, x' \in X\} = 4(X - X),$$

hence we conclude that

$$4(X - X) \supseteq B(S', \varepsilon),$$

or equivalently

$$X - X \supseteq B(4S', \varepsilon),$$

which contradicts the fact that $X - X$ does not contain a Bohr set. □

Proof of Theorem 2.2. Let $B = B(S, \varepsilon)$ be a Bohr set for which

$$d(B(S, \varepsilon) \setminus (A - A)) = 0;$$

the existence of which is Følner's theorem. Since this is an open set in the Bohr topology, there is a finite set T for which

$$B(S, \varepsilon) + T = \mathbb{Z}.$$

For $t \in T$ write $A_t = A \cap (B + t)$. Some of these sets have positive upper density; A' will be the union of these. Clearly $A \setminus A'$ is contained in the union of the finitely many A_t of density 0, so it has density 0 itself.

Put $B' = B(S, \varepsilon/3)$. We now show $A + A - A \supset A' + B'$. This is equivalent to $A + A - A \supset A_t + B'$ whenever $\overline{d}(A_t) > 0$.

Take arbitrary $a \in A_t$, $b \in B'$. Consider the set $a + b - A_t$. This has positive upper density and

$$a + b - A_t \subset A_t - A_t + B' \subset (B' + t) - (B' + t) + B' = B' + B' - B' \subset B.$$

Hence it is contained, up to a subset of density 0, in $A - A$, so we can find $a' \in A_t$ such that $a + b - a' \in A - A$, consequently $a + b \in a' + A - A \subset A + A - A$ as wanted.

□

In the rest of this section we give a third proof of Bergelson's theorem. In fact we can conclude that $A-A$ contains both an additive and a multiplicative structure.

Let $f:\mathbb{N}_+\mapsto\mathbb{N}_+$ be any function and $C\subseteq\mathbb{N}; C\neq\emptyset$. We will use the following notations:

$$FS(C)_f := \{\sum_{c_i\in X} w(i)c_i : X\subseteq C,\ |X|<\infty;\ 1\le w(i)\le f(i)\}.$$

Clearly $FS(C)\supseteq k\cdot C$ for every positive k. When X is the empty set then let the sum be zero.

Furthermore write

$$FP(C) := \{\prod_{c_i\in X} c_i : X\subseteq C;\ |X|<\infty\}.$$

Clearly we have

(2.1)
$$FS_f(\{c_1,c_2,\dots c_n\}) = FS_f(\{c_1,c_2,\dots c_{n-1}\}) + \{0,c_n,\dots,f(n)c_n\}.$$

and

$$FP(\{c_1,c_2,\dots c_n\}) = FP(\{c_1,c_2,\dots c_{n-1}\})\cdot\{1,c_n\} \tag{2.2}$$

for every $\{c_1,c_2,\dots c_n\}\subseteq\mathbb{N};\ n\ge 2$, or equivalently

$$FP(\{c_1,c_2,\dots c_n\}) = FP(\{c_1,c_2,\dots c_{n-1}\})\cup c_n\cdot FP(\{c_1,c_2,\dots c_{n-1}\}).$$

Theorem 2.3. *Let A be a set of integers, $\overline{d}(A)>0$. Let $f:\mathbb{N}_+\mapsto\mathbb{N}_+$ be any function. There exists an infinite set C of integers, such that*

$$A-A\supseteq FS(C)_f\cup FP(C).$$

Proof of Theorem 2.3. We start our proof by quoting Følner's theorem again. We have that there is a Bohr set for which the exceptional set has zero density, i.e. for some $B=B(S,\varepsilon)$, $E:=B(S,\varepsilon)\setminus(A-A)$, $d(E)=0$.

We will proof the existence of the infinite set C inductively.

Let $K_1:=f(1)$. Since any Bohr set has positive density and the exceptional set has zero density, furthermore by (1.2) one can find an element c_1 from $B(S,\varepsilon/K_1)$ such that $ic_1\notin E$, for $i=1,2,\dots K_1$. So we have

$$FS(\{c_1\})_f\cup FP(\{c_1\}) = \{0,c_1,\dots,K_1c_1\}\subseteq B\setminus E\subseteq A-A.$$

Assume now that the elements $c_1<c_2<\dots<c_n$ have been defined with the property

$$\mathcal{F}_n := FS(\{c_1,c_2,\dots c_n\})_f\cup FP(\{c_1,c_2,\dots c_n\})\subseteq B\setminus E\subseteq A-A.$$

Write $FP(\{c_1, c_2, \dots c_n\}) = \{p_1 < p_2 < \dots < p_m\}$, and let $K := \max\{f(n+1), p_m\}$. Define

$$\varepsilon_1 = \frac{1}{2K} \min\{\varepsilon - \|xs\| : x \in \mathcal{F}; s \in S\},$$

and let $B_1 := B(S, \varepsilon_1)$. Note that $B(S, \varepsilon_1) \subseteq B(S, \varepsilon)$. Observe that for every $i \leq K;\ i \in \mathbb{N}$ and every $s \in FS_f(\{c_1, c_2, \dots c_n\})$, and $c \in B_1$, we have $\|s + ic\| < \varepsilon$, hence

$$FS_f(\{c_1, c_2, \dots c_n\}) + \{0, c, 2c, \dots K \cdot c\} \subseteq B,$$

holds for $i = 1, 2, \dots K$.

Furthermore since E has zero density, it is easy to see that there exists a $B_1' \subseteq B_1$, $d(B_1') = d(B_1)$ and for every $c \in B_1'$,

$$FS_f(\{c_1, c_2, \dots c_n\}) + \{0, c, 2c, \dots K \cdot c\} \subseteq B \setminus E \subseteq A - A$$

also holds.

Observe, since $K \geq p_m$ and $0 \in FS_f(\{c_1, c_2, \dots c_n\})$ we also have

$$c \cdot \{p_1 < p_2 < \dots < p_m\} = c \cdot FP(\{c_1, c_2, \dots c_n\}) \subseteq$$

$$\subseteq \{0, c, 2c, \dots K \cdot c\} \subseteq B \setminus E \subseteq A - A.$$

Pick an arbitrary element $c_{n+1} := c \in B_1'$. For this element we obtain that

$$FS_f(\{c_1, c_2, \dots c_n, c_{n+1}\}) \subseteq$$

$$\subseteq FS_f(\{c_1, c_2, \dots c_n\}) + \{0, c, 2c, \dots K \cdot c\} \subseteq B \setminus E \subseteq A - A,$$

and

$$FP(\{c_1, c_2, \dots c_n\}) \cup c_{n+1} \cdot FP(\{c_1, c_2, \dots c_n\}) \subseteq B \setminus E \subseteq A - A$$

simultaneously holds. Thus we have that

$$\mathcal{F}_{n+1} \subseteq B \setminus E \subseteq A - A,$$

as we want.

So our desired set is

$$C := \{c_1 < c_2 < \dots c_n < \dots\}.$$

□

3. Further problems and results

We mention some open problems and some results without proof.

Bogolyubov's proof is effective: given the density of A one can specify k, η so that $A + A - A - A$ contains a Bohr k, η-set. Følner's proof is not effective, and the reason is that an effective version does not hold:

For every $\alpha < 1/2$, $k \in \mathbb{N}$ and $\eta > 0$ there is an $A \subset \mathbb{Z}$, $\overline{d}(A) > \alpha$ such that $\overline{d}(V \setminus (A - A)) > 0$ for every k, η-neighbourhood V.

Our Theorem 2.2 about $A + A - A$ applied Følner's, so the proof is not effective. We cannot decide whether an effective version holds. We can solve positively a related finite question. The result is as follows:

Let $\alpha > \varepsilon > 0$ be given. There are k, η depending on α and ε with the following property. For every $A \subset \mathbb{Z}_m$, $|A| \geq \alpha m$ the set $S = A + A - A - a$ contains a Bohr k, η-set for all but εm elements $a \in A$.

Here $\mathbb{Z}_m$ is the group of residues modulo m and Bohr sets are defined as in (1.1) with the modification that for s only rational numbers of the form k/m can be used.

Assume $\overline{d}(A) > 0$. Is $A - A$ a Bohr neighbourhood of *something*? We know it may not be a neighbourhood of 0, and 0 is the most natural difference. For the analogous finite question we can give a negative answer, which is as follows:

Let $\alpha < 1/2$, k, η be given. For all large m there is an $A \subset \mathbb{Z}_m$, $|A| \geq \alpha m$ such that $A - A - x$ does not contain a Bohr k, η-set for any $x \in \mathbb{Z}_m$.

Proofs of these results will be published elsewhere.

Is $A - A$ a Bohr neighbourhood of 0 under the stronger assumption that A has positive lower Banach density (A is syndetic, has bounded gaps)?

Here we cannot solve the related finite problem either, and do not have any heuristic reasoning in any direction.

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