ANSWER TO A QUESTION OF BURR AND ERDŐS ON RESTRICTED ADDITION AND RELATED RESULTS

N. HEGYVÁRI, F. HENNECART AND A. PLAGNE

Abstract. We study the gaps in the sequence of sums of h pairwise distinct elements of a given sequence $\mathcal{A}$ in relation with the gaps in the sequence of sums of h not necessarily distinct terms of $\mathcal{A}$. We present several results on this topic. One of them gives a negative answer to a question by Burr and Erdős.

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1. Introduction

In [1], Erdős states the following:

> Here is a really recent problem of Burr and myself : An infinite sequence of integers $a_1 < a_2 < \cdots$ is called an asymptotic basis of order k, if every large integer is the sum of k or fewer of the a's. Let now $b_1 < b_2 < \cdots$ be the sequence of integers which is the sum of k or fewer distinct a's. Is it true that
>
> $$\limsup(b_{i+1} - b_i) < \infty.$$
>
> In other words the gaps between the b's are bounded. The bound may of course depend on k and on the sequence $a_1 < a_2 < \cdots$.

For $h \geq 1$, we will use the following notation for addition and restricted addition, according to which $h\mathcal{A}$ will denote the set of the sums of h not necessarily distinct elements of $\mathcal{A}$, and $h \times \mathcal{A}$, the set of the sums of h pairwise distinct elements of $\mathcal{A}$.

If $\mathcal{A}$ is an increasing sequence $a_1 < a_2 < \cdots$, the largest asymptotic gap in $\mathcal{A}$, that is

$$\limsup_{i\to+\infty}(a_{i+1} - a_i)$$

is denoted $\Delta(\mathcal{A})$.

We shall write $\mathcal{A} \sim \mathbb{N}$ to denote that a set of integers $\mathcal{A}$ contains all but finitely many natural integers. According to Erdős-Burr definition, a set $\mathcal{A}$ is an asymptotic basis of order k if k is the least integer such that $\bigcup_{j=1}^{k} j\mathcal{A} \sim \mathbb{N}$, or equivalently that $h(A \cup \{0\}) \sim \mathbb{N}$.

The question of Burr and Erdős takes the shorter form: is it true that if $h(\{0\} \cup \mathcal{A}) \sim \mathbb{N}$, then

$$\Delta(\mathcal{A} \cup 2 \times \mathcal{A} \cup \cdots \cup h \times \mathcal{A}) < +\infty\,?$$

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We may also ask the following even more natural question: is it true that $\Delta(h\mathcal{A}) < +\infty$ implies $\Delta(h \times \mathcal{A}) < +\infty$? This would imply (and thus give another proof of) the main result in [5] which states that if $\mathcal{A}$ is an asymptotic basis of order h, then $h \times \mathcal{A}$ has a positive lower density, as it was conjectured in [2].

We will show that the answer to both questions is no, except if $h = 2$:

Theorem 1. *If $(\mathcal{A} \cup 2\mathcal{A}) \sim \mathbb{N}$ then*

$$\Delta(\mathcal{A} \cup 2 \times \mathcal{A}) \leq 2.$$

If $2\mathcal{A} \sim \mathbb{N}$ then $\Delta(2 \times \mathcal{A}) \leq 2$.

Let $h \geq 3$. There exists a set $\mathcal{A}$ such that $h(\{0\} \cup \mathcal{A}) \sim \mathbb{N}$ and

$$\Delta(\mathcal{A} \cup 2 \times \mathcal{A} \cup \cdots \cup h \times \mathcal{A}) = +\infty.$$

There exists a set $\mathcal{A}$ such that $h\mathcal{A} \sim \mathbb{N}$ and $\Delta(h \times \mathcal{A}) = +\infty$.

Concerning the case of asymptotic bases of order 2, we know that there exist such bases whose restricted (asymptotic) order is not equal to 2 (see [7] and [6] for more details): we say that for any (asymptotic) basis $\mathcal{A}$, the restricted order of $\mathcal{A}$, if it exists, is the least integer h such that any large enough integer is the sum of h or fewer pairwise distinct elements of $\mathcal{A}$. We denote it by $\mathrm{ord}_r(\mathcal{A})$. It follows that $\Delta(\mathcal{A} \cup 2 \times \mathcal{A}) \geq 2$ is possible for bases $\mathcal{A}$ of order 2.

The next natural question is then : assume that $h\mathcal{A} \sim \mathbb{N}$, that is $h\mathcal{A}$ contains all but finitely many positive integers. Is it true that there exists an integer k such that $\Delta(k \times \mathcal{A}) < +\infty$? If so k could depend on $\mathcal{A}$. But, suppose k exists: is this value of k uniformly (with respect to $\mathcal{A}$) bounded from above (in term of h)? If so, write $k(h)$ for the maximal possible value:

$$k(h) = \max_{h\mathcal{A} \sim \mathbb{N}} \min\{k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) \text{ is finite}\}.$$

It is easily seen that $k(2) = 2$. No other value of $k(h)$ is known.

Conjecture 1. *The function $k(h)$ is well defined in the sense that it is finite.*

If this conjecture is true, what is the asymptotic behavior of $k(h)$? We will construct a counterexample to the Erdős-Burr conjecture. More precisely we will prove the following lower bound which obviously implies Theorem 1 for $h \geq 3$.

Theorem 2. *Let $h \geq 2$. We have*

$$k(h) \geq 2^{h-2} + h - 1.$$

This question is closely related to the following problem : If $\mathcal{A}$ is an (asymptotic) basis of order h which possesses a restricted order $\mathrm{ord}_r(\mathcal{A})$, is it true that $\mathrm{ord}_r(\mathcal{A})$ is bounded in terms of h. If so let us define $f(h)$ the maximal possible value taken by $\mathrm{ord}_r(\mathcal{A})$, when $\mathcal{A}$ runs over the bases of order h having a finite restricted order. For this problem, we may reuse the counterexample quoted above.

Theorem 3. *One has*

$$f(h) \geq 2^{h-2} + h - 1.$$

In another direction, we can study, for a given set of natural integers $\mathcal{A}$, the asymptotic behavior of the sequence $(\Delta(h \times \mathcal{A}))_{h \geq h_0}$. The first observation is that this sequence is well-defined for some h_0 as far as $\Delta(h_0 \times \mathcal{A})$ is finite. Indeed we have

Proposition 4. *Let $\mathcal{A}$ be a given set of integers. Assume that $\Delta(h_0 \times \mathcal{A})$ is finite for some integer h_0, then for any $h \geq h_0$, $\Delta(h \times \mathcal{A})$ is finite.*

This result implies that

$$k(h) = 1 + \max_{h\mathcal{A} \sim \mathbb{N}} \max\{k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) = +\infty\}.$$

According to what obviously happens in the case of the usual addition, it would be of some interest to show that more precisely, for any given set $\mathcal{A}$, the function $\Delta(h \times \mathcal{A})$ is non-increasing :

Conjecture 2. *Let $\mathcal{A}$ be any set of integers, then the function $\Delta(h \times \mathcal{A})$ is non-increasing.*

We will observe firstly the following:

Proposition 5. *Let $\mathcal{A}$ be any set of integers, then*

$$\Delta(3 \times \mathcal{A}) \leq \Delta(2 \times \mathcal{A}).$$

More interestingly, we will show the following partial result:

Theorem 6. *Let $\mathcal{A}$ be a set of integers. Then there exists an increasing sequence of integers $(h_j)_{j\geq 1}$ such that $(\Delta(h_j \times \mathcal{A}))_{j\geq 1}$ is non-increasing.*

The above result clearly implies that $\Delta((h+1)\times\mathcal{A}) \leq \Delta(h\times\mathcal{A})$ for infinitely many positive integers h. Theorem 6 is a direct consequence of

Theorem 7. *Let $\mathcal{A}$ be a set of integers and h be the least positive integer such that $\Delta(h\times\mathcal{A})$ is finite. Then there exists an increasing sequence of integers $(h_j)_{j\geq 0}$ with $h_0 = h$ such that for any $j \geq 1$, one has $h_{j+1} \leq h_j + h + 1$ and $\Delta(h_{j+1} \times \mathcal{A}) \leq \Delta(h_j \times \mathcal{A})$.*

This shows that $\Delta((h+1) \times \mathcal{A}) \leq \Delta(h \times \mathcal{A})$ for any h belonging to some set of positive integers having a positive lower density, bounded from below by $1/(h+1)$.

Suppose we have proved that $k(h)$ is finite, we could try to prove the same kind of result under the only assumption that $\underline{\mathrm{d}}h\mathcal{A} > 0$ (instead of $h\mathcal{A} \sim \mathbb{N}$). We may prove that for any set of integers $\mathcal{A}$ such that $\underline{\mathrm{d}}h\mathcal{A} > 0$, there exists an integer k such that $\Delta(k \times \mathcal{A})$ is finite. Clearly this result, if true, could not be uniform in $\mathcal{A}$. The smallest $\underline{\mathrm{d}}h\mathcal{A}$ is, the largest k should be. Let us introduce for $\beta > 0$

$$k_1(\beta, h) = \max_{\underline{\mathrm{d}}h\mathcal{A}\geq\beta} \min\left\{k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) \text{ is finite}\right\}.$$

In fact, we can prove the following result which shows that k_1 is as well defined as k, in some sense.

Theorem 8. *Let β such that $0 < \beta \leq 1$. Assume $k(h)$ is finite for any $h \geq 1$. Then*

$$k_1(\beta, h) \leq k\left(\left\lceil \left(1+\frac{1}{h}\right)\frac{1}{\beta}\right\rceil h\right).$$

where $\lceil u \rceil$ is the ceiling of u.

All of these results emerge some new questions in the area of restricted addition. We consider this paper as a preliminary investigation on this type of problems.

2. The proofs

For any real numbers x, y, $[x, y]$ and $[x, y)$ will denote the set of all integers n such that $x \leq n \leq y$ and $x \leq n < y$ respectively.

Proof of Theorems 1, 2 and 3. Let us first consider the case $h = 2$. Clearly the odd elements in $2\mathcal{A}$ do belong to $2 \times \mathcal{A}$. This implies that if $2\mathcal{A} \sim \mathbb{N}$, then $\Delta(2 \times \mathcal{A}) \leq 2$. This also implies that the odd elements in $\mathcal{A} \cup 2\mathcal{A}$ are in $\mathcal{A} \cup (2 \times \mathcal{A})$. This shows that $\mathcal{A} \cup 2\mathcal{A} \sim \mathbb{N}$ implies $\Delta(\mathcal{A} \cup (2 \times \mathcal{A})) \leq 2$.

In the case $h \geq 3$, it is enough to provide a counterexample.

For $k \geq 0$, we put

$$\mathcal{A}_k = [0, x_k^2] \cup \left\{2^j x_k^2 \,:\, j = 1, 2, \ldots, h-2\right\},$$

where $x_0 = 1$ and $x_{k+1} = (3 \cdot 2^{h-2} - 1)x_k^2 + hx_k$, $k \geq 0$. Then let

$$\mathcal{A} = \{0\} \cup \bigcup_{k \geq 0} (x_k + \mathcal{A}_k).$$

Any integer in $[0, (2^{h-1} - 1)x_k^2]$ can be written as a sum of $h - 1$ elements of $\mathcal{A}_k$, thus $[0, (3 \cdot 2^{h-2} - 1)x_k^2] \subset h\mathcal{A}_k$ yielding $[x_k, x_{k+1}] \subset h((x_k + \mathcal{A}_k) \cup \{0\})$. It follows that $\mathcal{A}$ is a basis of order at most h.

Concerning restricted addition, we see that for $l \geq h - 2$, we have

$$\max(l \times \mathcal{A}_k) \leq (2^{h-1} - 2)x_k^2 + (l - h + 2)x_k^2 = (2^{h-1} + l - h)x_k^2,$$

hence

$$x_{k+1} - \max(l \times (x_k + \mathcal{A}_k)) \geq (2^{h-2} - l + h - 1)x_k^2 + (h - l)x_k.$$

If $l \leq 2^{h-2} + h - 2$, then $x_{k+1} - \max(l \times (x_k + \mathcal{A}_k)) \geq x_k^2 - (2^{h-2} - 2)x_k$ which tends to infinity as k tends to infinity. It follows that $k(h) \geq 2^{h-2} + h - 1$, as asserted in Theorem 2.

It is easily seen that the basis $\mathcal{A}$ of order h given above has restricted order $2^{h-2} + h - 1$, yielding Theorem 3. □

Proof of Proposition 4. We denote by $a_1 < a_2 < \cdots$ the elements of $\mathcal{A}$. Let $x \in h \times \mathcal{A}$ be larger than $a_1 + a_2 + \cdots + a_h$. Then there exists $i = i(x)$ among $1, 2, \ldots, h$ such that $x \in h \times (\mathcal{A} \smallsetminus \{a_i\})$. It follows that $a_i + x \in (h+1) \times \mathcal{A}$. Thus clearly $\Delta((h+1) \times \mathcal{A}) \leq \Delta(h \times \mathcal{A}) + a_h - a_1$. □

Proof of Theorem 5. Let $X = \{x_1 < x_2 < \cdots < x_k < \cdots\}$ be a set of natural integers. We denote $D(X) = \max(x_{k+1} - x_k)$ and recall $\Delta(X) = \limsup_{k\to+\infty}(x_{k+1} - x_k)$.

Let $d > 0$. We shall say that X d-covers an interval I if the union of the balls centered on the elements of X with radius $d/2$ contains I. In other words :

$$\forall r \in I, \exists x \in X \text{ tel que } |x - r| \leq d/2.$$

Let $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$. Assume $\Delta(2 \times \mathcal{A}) = d < +\infty$. There exists an x_0 such that $[x_0, +\infty)$ is d-covered by $2 \times \mathcal{A}$. We shall see that for any $a_i \in \mathcal{A}$ large enough, the interval $[a_i + x_0, a_{i+1} + x_0)$ is d-covered by $3 \times \mathcal{A}$.

First case : if $a_{i+1} \leq 2a_i - x_0 - d/2$, then $a_i + (2 \times \mathcal{A}) \cap [0, a_i)$ is contained in $3 \times \mathcal{A}$ and d-covers $[a_i + x_0, 2a_i - d/2)$ which contains $[a_i + x_0, a_{i+1} + x_0)$.

Second case : if $a_{i+1} > 2a_i - x_0 - d/2$, then

$$(2 \times \mathcal{A}) \cap [3a_i/2, a_{i+1}) \subset 2 \times (\mathcal{A} \cap [a_i/2, a_i]).$$

Let $a \in \mathcal{A}$ such that $d/2 + x_0 < a < a_i/2 - d$ (we may always find such an a if a_i is large enough). Then

$$a + (2 \times \mathcal{A}) \cap [3a_i/2, a_{i+1}) \subset 3 \times \mathcal{A}.$$

Since $[3a_i/2, a_{i+1})$ is d-covered by $2 \times \mathcal{A}$, the interval $[3a_i/2 + d/2 + a, a + a_{i+1} - d/2)$ is d-covered by $3 \times \mathcal{A}$. Thus $[2a_i - d/2, a_{i+1} + x_0)$ is d-covered by $3 \times \mathcal{A}$.

Since $[a_i+x_0, 2a_i-d/2)$ is d-covered by $a_i+(2\times\mathcal{A})\cap[0, a_i)$ we conclude that $[a_i+x_0, a_{i+1}+x_0]$ is d-covered by $3 \times \mathcal{A}$. □

Proof of Theorem 7. Let $\mathcal{A}$ such that $\Delta = \Delta(h \times \mathcal{A}) < +\infty$. This implies that for any sufficiently large x,

$$\mathcal{A}(x) := |\mathcal{A} \cap [1, x]| \geq Cx^{1/h},$$

for some constant C depending only on Δ. Now the number of subsets of $\mathcal{A} \cap [0, x]$ with cardinality $h + 1$ is the binomial coefficient $\binom{\mathcal{A}(x)}{h+1} \gg x^{1+1/h}$ where the implied constant depends on both $\mathcal{A}$ and h. Choose an x such that $\binom{\mathcal{A}(x)}{h+1} \geq h!h^{h+1}x$. It thus exists an integer n less than $(h + 1)x$ such that

$$n = a_1^{(i)} + \cdots + a_{h+1}^{(i)}, \quad i = 1, \ldots, h!h^h,$$

where the $h!h^h$ sets of $h+1$ pairwise distinct elements of $\mathcal{A}$, $E_i := \{a_1^i, \ldots, a_{h+1}^i\}$ are distinct. By a deep result due to Erdős and Rado on Δ-systems (cf. [3]), we know that there are $h+1$ sets E_{i_j}, $j = 1, \ldots, h+1$, and a set F of cardinality denoted by $|F|$ such that $E_{i_j} \cap E_{i_k} = F$, for any $1 \leq j \neq k \leq h+1$. It follows that the integer

$$n' = n - \sum_{a\in F} a$$

can be written as a sum of $h+1-|F|$ pairwise distinct elements of $\mathcal{A}$ in at least $h+1$ ways, such that all the elements appearing in any representation of $\mathcal{A}$ in $(h+1-|F|)\times\mathcal{A}$ are pairwise distinct. This shows that

$$n'+(h\times\mathcal{A})\subset(2h+1-|F|)\times\mathcal{A}.$$

and finally $\Delta(h\times\mathcal{A})=\Delta(n'+(h\times\mathcal{A}))\geq\Delta(h_1\times\mathcal{A})$, where $h_1=2h+1-|F|$. Iterating this process, we get an increasing sequence $(h_i)_{i\geq 0}$, where $h_0=h$, such that $\Delta(h_i\times\mathcal{A})=\Delta(n'+(h_i\times\mathcal{A}))\geq(h_{i+1}\times\mathcal{A})$, where $h_{i+1}=h_i+h+1-|F|\leq h_i+h+1$. □

Proof of Theorem 8. Let h be a positive integer and $\mathcal{A}$ be a sequence of integers. We put $\mathcal{B}=h\mathcal{A}$ and assume that $\underline{\mathrm{d}}\mathcal{B}\geq\beta>0$. Define

$$j=\left\lceil\left(1+\frac{1}{h}\right)\frac{1}{\beta}\right\rceil.$$

We thus have

$$j\underline{\mathrm{d}}\mathcal{B}>1\geq\underline{\mathrm{d}}j\mathcal{B}.$$

By Kneser's theorem on asymptotic densities of sequences of integers (cf. [9, 10], [4] or [12]), we obtain that there exist an integer $g\geq 1$ and a sequence $\mathcal{B}_1$ of integers such that

$$\mathcal{B}\subseteq\mathcal{B}_1,\quad g+\mathcal{B}_1\subset\mathcal{B}_1,\quad j\mathcal{B}_1\smallsetminus j\mathcal{B}\text{ is finite,}$$

and

$$\underline{\mathrm{d}}j\mathcal{B}_1\geq j\underline{\mathrm{d}}\mathcal{B}_1-\frac{j-1}{g}.$$

We may assume that g is the least integer having this property. This gives

$$g\leq\frac{j-1}{j\beta-1},$$

hence $g\leq(j-1)h\leq jh$. We denote by $\overline{\mathcal{A}}\subset\mathbb{Z}/g\mathbb{Z}$ the image of $\mathcal{A}$ by the canonical homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}/g\mathbb{Z}$, the group of residue classes modulo g. Let H be the period of $g\overline{\mathcal{A}}$, that is the subgroup of $\mathbb{Z}/g\mathbb{Z}$ formed by the elements c such that $c+g\overline{\mathcal{A}}=g\overline{\mathcal{A}}$. Since $g\leq jh$, $jh\overline{\mathcal{A}}=j\overline{\mathcal{B}}=j\overline{\mathcal{B}_1}$ satisfies

$$j\overline{\mathcal{B}_1}+H=j\overline{\mathcal{B}_1}.$$

By minimality of g, we get $H=\{0\}$, thus by Kneser's Theorem on the addition of sets in an abelian group (see [8] or [11]), we deduce

$$g\geq|g\overline{\mathcal{A}}|\geq g(|\overline{\mathcal{A}}|-1)+1,$$

yielding $|\overline{\mathcal{A}}|=1$. Let $a+gx_i$, $i\geq 1$, be the elements of $\mathcal{A}$ and $\mathcal{A}_1$ the set of the x_i's. From the identity $jh\mathcal{A}=j\mathcal{B}$, we get $jh\mathcal{A}_1\sim\mathbb{N}$. It follows that $\Delta(k(jh)\times\mathcal{A}_1)$ is finite, and accordingly $\Delta(k(jh)\times\mathcal{A})<+\infty$. □

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Norbert Hegyvári
Department of Mathematics
Eötvös University
Budapest, Pázmány P. st. 1/C
P.O. Box 120
H-1518 Budapest
Hungary
E-mail address: `norb@ludens.elte.hu`

François Hennecart
LaMUSE
Université de Saint-Étienne
42023 Saint-Étienne Cedex 2
France
E-mail address: `francois.hennecart@univ-st-etienne.fr`

Alain Plagne
CMLS
UMR 7640 du CNRS
École polytechnique
91128 Palaiseau Cedex
France
E-mail address: `plagne@math.polytechnique.fr`