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Note

On additive and multiplicative Hilbert cubes

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Abstract

Given subset E of natural numbers $FS(E)$ is defined as the collection of all sums of elements of finite subsets of E and any translation of $FS(E)$ is said to be Hilbert cube. We can define the multiplicative analog of Hilbert cube as well. E.G. Strauss proved that for every $\varepsilon > 0$ there exists a sequence with density $> 1 - \varepsilon$ which does not contain an infinite Hilbert cube. On the other hand, Nathanson showed that any set of density 1 contains an infinite Hilbert cube. In the present note we estimate the density of Hilbert cubes which can be found avoiding sufficiently sparse (in particular, zero density) sequences. As a consequence we derive a result in which we ensure a dense additive Hilbert cube which avoids a multiplicative one.

Keywords: Hilbert cube; *IP*-set

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers. We call $H \subset \mathbb{N}$ a k-dimensional Hilbert cube if there exist $a > 0$ and $x_1 < x_2 < \cdots < x_k$ such that

$$H = H(a, x_1, \ldots, x_k) = \left\{ a + \sum_{i=1}^{k} \varepsilon_i x_i \colon \ \varepsilon_i = 0 \text{ or } 1 \right\}.$$

The name is a reverence to Hilbert who proved essentially the first Ramsey-type theorem: for any $k \geqslant 1$, if $\mathbb{N}$ is finitely colored than there exists in one color infinitely many translates of a k-cube. The density version of Hilbert's result, which was an important part of Szemerédi's proof of his

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theorem on arithmetic progressions, is the following: Let A be an infinite sequence of integers with

$$d(A) = \lim_{n\to\infty} \inf A(n)/n > 0,$$

where $A(n) = \sum_{a_i \leqslant n} 1$. Then there exists a β so that for every n, $A \cap [1, n]$ contains a k-cube with

$$k > \beta \log\log n. \tag{1.1}$$

The author investigated how sharp this bound is in certain sets (see [1,2]). Related questions are discussed in [3,4]. We can define *infinite* Hilbert cubes as well: Let $x_1 < x_2 < \cdots$ be an infinite sequence of integers. An infinite Hilbert cube with base point a_0 and vertices $x_1 < x_2 < \cdots$ is defined by

$$H = H(a_0, x_1, \ldots) = \left\{ a_0 + \sum_{i=1}^{\infty} \varepsilon_i x_i\colon\ \varepsilon_i = 0 \text{ or } 1;\ \sum \varepsilon_i < \infty \right\}.$$

As in (1.1) we may estimate the number of vertices not exceeding n of an infinite Hilbert cube.

Let $H = H(a_0, x_1, x_2, \ldots)$ be a Hilbert cube. We define the dimension function of H as

$$H(n) = \bigl|\{x_1 < x_2 < \cdots\} \cap [1, n]\bigr|.$$

In [5] the authors proved that the sequence of squarefree numbers contains an infinite H-cube. They derived this result from the following: Assume A is an infinite sequence of integers not containing 1 and $\sum \frac{1}{a} < \infty$; define $B(A)$ as the set of natural numbers that are divisible by an element of A. Then $B^c(A)$, the complement of $B(A)$ contains an infinite H-cube.

In [6] the author proved the existence of infinite H-cubes which avoid $B(P')$ where P' is a subsequence of primes. This problem can be transformed to the following problem; find an H-cube in $B^c(P')$ (in the complement of $B(P')$). Clearly $B^c(P')$ consists of all integers composed solely of the primes $P'' := \{2, 3, 5, \ldots\} \setminus P'$, in other words $m \in B^c(P')$ if and only if $m = \prod_{p \in P''} p^{\alpha_p}$.

In the ergodic-combinatorial number theory a set is said to be $\overline{IP}$-set if it contains an $\overline{FS(A)}$ where $\overline{FS(A)}$ is the set of all sums of the form $\sum x_i a_i$; x_i is an integer fulfils $0 \leqslant x_i \leqslant i$ and $\sum x_i < \infty$. We write $FS(A)$ if $x_i \in \{0, 1\}$ is also required.

A multiplicative generalization of this notion would be the following: a set E is said to be $\overline{IP_\Pi}$-set if it contains an $\overline{FP(E)}$ where $\overline{FP(E)}$ is the set of all products of the form $\prod e_i^{\alpha_i}$; $e_i \in E$. We have $\overline{FP(E)} = B^c(A)$ when E collects all primes p, $p \nmid a$, for all $a \in A$.

In the present note we investigate the above mentioned problem when P is a lacunary sequence of primes.

Definition. A sequence $P = \{p_1 < p_2 < \cdots < p_i < \cdots\}$ is said to be λ-lacunary if for every $i = 1, 2, \ldots$

$$\frac{p_{i+1}}{p_i} > \lambda > 1.$$

The main aim of this note to estimate the dimension $H_\lambda(n)$ of a Hilbert cube which avoids $\overline{FP(P)}$. This will be dealt with Section 3.

2. Hilbert cube in dense sets

In [7] E.G. Strauss proved that for every $\varepsilon > 0$ there exists a sequence with density $> 1 - \varepsilon$ which does not contain an infinite Hilbert cube. On the other hand, it was proved in [8] that every sequence of integers with density 1 contains an infinite Hilbert cube.

Let us start with two remarks. Firstly note that for a given interval $I = [a, a + m]$ if an H-cube $H(a_0, x_1 < \cdots < x_s)$ lies in I then clearly $s < c\sqrt{m}$. Secondly if for some $A \subseteq [1, N]$, we would like to avoid A by an H-cube, then statistically we have a gap with size $\frac{N}{|A|}$ and by the previous remark there is a cube with $H(N) \sim c\sqrt{\frac{N}{|A|}}$. This argument works just in a finite case and completely false in the infinite case.

However in the next theorem we will show that essentially apart from a $\log n$ factor a same conclusion remains true.

Theorem 2.1. *Let A be a sequence of integers and let $\omega : \mathbb{N} \to \mathbb{R}^+$ be any function and assume that $\omega(x) \to \infty$ as $x \to \infty$. There exists a cube H for which*

$$\limsup_{n\to\infty} \frac{H(n)}{\sqrt{n/(A(n)\cdot\omega(n)\cdot\log^2 n)}} > 0. \tag{2.1}$$

Proof of Theorem 2.1. We have to find a sequence $n_1 < n_2 < \cdots < n_k < \cdots$, a Hilbert cube H, satisfying

$$H(n_k) > c\sqrt{\frac{n_k}{A(n_k)\cdot\omega(n_k)\cdot\log^2 n_k}}.$$

Let $a_0 = 0$ and let $x_1 = \min\{x\colon\ 0 < x \notin A\}$, $n_1 = \min\{x\colon\ 0 < x \in A\}$, and assume that the sequences $\{x_1, x_2, \ldots, x_m\}$ and $n_1 < n_2 < \cdots < n_t$ have been defined so that

$$\bigl|\{x_1, x_2, \ldots, \} \cap [1, n_k]\bigr| > c\sqrt{\frac{n_k}{A(n_k)\cdot\omega(n_k)\cdot\log^2 n_k}}, \tag{2.2}$$

and let the constant be so small that (2.2) holds for $k = 1$ as well ($A(n_1) = 1$). Let $M = \sum_{i=1}^m x_i$. Define n_{t+1} the smallest integer n for which $\omega(n) > 2M + 1$. Since the function $\omega(x)$ tends to infinity as x tends to infinity such a number exists. Furthermore assume without loss of generality that $\omega(x) < x/2$.

Let

$$A \cap [1, n_{t+1}] = \{a_1 < a_2 < \cdots < a_r\},$$

and define the set B by

$$B = \bigcup_{i=1}^{r} [a_i - M, a_i + M].$$

Let $n \in B \cap \mathbb{N}$ and write n in a canonical form $n = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_z^{\alpha_z}$. Clearly

$$2^z \leqslant p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_z^{\alpha_z} \leqslant n_{t+1} + M < 2n_{t+1},$$

hence

$$z \leqslant \frac{\log 2n_{t+1}}{\log 2}.$$

Now set

$$\widetilde{P} = \{p_i\colon\ \exists b \in B\, p_i \mid b;\ \ p_i \text{ is a prime number}\}.$$

Clearly

$$|\widetilde{P}| \leqslant |B| \cdot \max z \leqslant r \cdot (2M+1) \cdot \frac{\log 2n_{t+1}}{\log 2} \leqslant A(n_{t+1}) \cdot \omega(n_{t+1}) \cdot \frac{\log 2n_{t+1}}{\log 2}.$$

Now consider the smallest prime number not contained in the set $\widetilde{P}$. Let it be p_L. We have

$$L \leqslant |\widetilde{P}| + 1 \leqslant 1.5A(n_{t+1}) \cdot \omega(n_{t+1}) \cdot \frac{\log 2n_{t+1}}{\log 2} < 3A(n_{t+1}) \cdot \omega(n_{t+1}) \cdot \log 2n_{t+1}.$$

Finally define

$$x_{m+i} = i \cdot p_L;$$

for $i = 1, 2, \dots, \lfloor\sqrt{\frac{n_{t+1}}{p_L}}\rfloor$. Let $T = \lfloor\sqrt{\frac{n_{t+1}}{p_L}}\rfloor$. Let $H_m = H(a_0, x_1, \dots, x_{m+T})$. Clearly

$$\max_{h \in H_m} h < p_L \cdot \frac{T^2}{2} + M \leqslant n_{t+1},$$

since $p_L \cdot \frac{T^2}{2} \leqslant \frac{n_{t+1}}{2}$, and $M < \omega(n_{t+1}) < n_{t+1}/2$.

Thus we have $m + T$ many vertices of our Hilbert cube, i.e.

$$H(n_{t+1}) = m + T \geqslant \sqrt{\frac{n_{t+1}}{p_L}}.$$

By the prime number theorem $p_L \sim L \cdot \log L$, and so

$$\begin{aligned} H(n_{t+1}) &\geqslant \frac{1}{2}\sqrt{\frac{n_{t+1}}{L \cdot \log L}} \geqslant \frac{1}{2}\sqrt{\frac{n_{t+1}}{6A(n_{t+1}) \cdot \omega(n_{t+1}) \cdot \log^2 n_{t+1}}} \\ &> c\sqrt{\frac{n_{t+1}}{A(n_{t+1}) \cdot \omega(n_{t+1}) \cdot \log^2 n_{t+1}}}, \end{aligned}$$

if $c < \frac{1}{6}$.

In the rest of the proof we have to show that $H = H(0, x_1 < x_2 < \dots < x_{m+T})$ avoids the sequence A. Consider first the Hilbert cube $H' = H'(0, x_{m+1} < x_{m+2} < \dots < x_{m+T})$. Since every i, $1 \leqslant i \leqslant T$ $p_L \mid x_i$ hence for every $k \in H'$, $p_L \mid k$. Furthermore by the definition of p_L for every n for which $\min_{a \in A} |a - n| \leqslant M$ holds, $p_L \nmid n$. Thus we have for every $a \in A$ and $k \in H'$

$$|a - k| > M = \sum_{i=1}^{m} x_i.$$

It follows, using

$$\begin{aligned} H &= H(0, x_1 < x_2 < \dots < x_{m+T}) \\ &= H(0, x_1 < x_2 < \dots < x_m) + H'(0, x_{m+1} < x_{m+2} < \dots < x_{m+T}), \end{aligned}$$

that for every $k \in H(0, x_1 < x_2 < \dots < x_{m+T})$ and for every $a \in A$ that

$$|a - k| \geqslant 1,$$

as we stated. □

3. Additive and multiplicative Hilbert cubes

In the present section we give an estimation for the dimension of an additive Hilbert cube which avoids a multiplicative one. We prove

Theorem 3.1. *Let $P = \{p = p_1 < p_2 < \cdots < p_i < \cdots\}$ be a λ-lacunary sequence of primes. There exists an infinite Hilbert cube $H_\lambda = H(a_0, x_1, x_2, \ldots)$ such that $H_\lambda \cap \overline{FP(P)} = \emptyset$, and*

$$\limsup_{n\to\infty} \frac{H_\lambda(n)\cdot e^{c'\sqrt{\log n}}}{\sqrt{n}} > 0,$$

for all $c' > \pi\sqrt{\frac{2}{3\log\lambda}}$.

Proof of Theorem 3.1. Since the sequence of primes P is λ-lacunary we have $p_{i+1} > \lambda p_i$ for every $i \geqslant 1$. We conclude from it that for every $i \geqslant 1$,

$$p_{i+1} > \lambda^i p \tag{3.1}$$

holds.

We need the following lemma:

Lemma 3.2. *If the sequence $P = \{p = p_1 < p_2 < \cdots < p_i < \cdots\}$ is a λ-lacunary sequence of primes then*

$$\left|\overline{FP(P)} \cap [1,n]\right| < c_\lambda \frac{e^{c\sqrt{\log n}}}{\sqrt{\log n}}, \tag{3.2}$$

where the constants c and c_λ depend only on λ (c can be chosen as $c = \pi\sqrt{\frac{2}{3\log\lambda}}$).

Proof of Lemma 3.2. By (3.1) for $p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k}$ we have

$$p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k} > \lambda^{\alpha_1}\cdot p^{\alpha_1}\cdot\lambda^{2\alpha_2}\cdot p^{\alpha_2}\cdot\ldots\cdot\lambda^{k\alpha_k}\cdot p^{\alpha_k} = \lambda^{\sum\alpha_i\cdot i}\cdot p^{\sum\alpha_i}.$$

Now let us define K by $\lambda^K \leqslant n$. Since

$$\left\{(\alpha_1,\alpha_2,\ldots,\alpha_k)\colon\ p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k} \in \overline{FP(P)}\cap[1,n]\right\}$$
$$\subseteq \Lambda := \left\{(\alpha_1,\alpha_2,\ldots,\alpha_k)\colon\ \sum\alpha_i\cdot i \leqslant K\right\},$$

thus we only have to give an upper estimation for the cardinality of Λ.

We shall use a classical result on partitions. Denote by $p(n)$ the unrestricted partition function which counts the number of partitions $n = \sum a_i$; $(a_1 \geqslant a_2 \geqslant \cdots)$. Then by the theorem of Ramanujan (see in [9]) we have

$$p(n) \sim \frac{e^{c_0\sqrt{n}}}{4\sqrt{3}n} \tag{3.3}$$

($c_0 = \pi\sqrt{2/3}$).

An easy calculation shows that the summation function of $p(n)$ is

$$P(K) = \sum_{k\leqslant K} p(k) \sim \frac{e^{c_0\sqrt{K}}}{2\pi\sqrt{2}\sqrt{K}}.$$

Thus we obtain

$$\left|\overline{FP(P)} \cap [1,n]\right| \leqslant |\Lambda| \leqslant \frac{e^{c_0\sqrt{K}}}{2\pi\sqrt{2}\sqrt{K}} = c_\lambda \frac{e^{c\sqrt{\log n}}}{\sqrt{\log n}},$$

where $c = \frac{c_0}{\sqrt{\log\lambda}} = \pi\sqrt{\frac{2}{3\log\lambda}}$, as we claimed. □

Now let us use Theorem 2.1 for the sequence $\overline{FP(P)}$, i.e. let A be the sequence $\overline{FP(P)}$. Choose the function ω to $\omega(n) := e^{c''\sqrt{\log n}}$ ($c'' > 0$). We have

$$A(n)\cdot\omega(n)\cdot\log n\cdot\log\bigl(A(n)\cdot\omega(n)\cdot\log n\bigr) < c_\lambda e^{c'\sqrt{\log n}},$$

where $c' > c + c''$. Thus by Theorem 2.1 we conclude that there exists an infinite Hilbert cube H_λ which avoids the sequence $A = \overline{FP(P)}$, and for which

$$\limsup_{n\to\infty} \frac{H_\lambda(n)\cdot e^{c'\sqrt{\log n}}}{\sqrt{n}} > 0$$

($c' > \pi\sqrt{\frac{2}{3\log\lambda}}$) as we stated. □

The author acknowledges a suggestion of J. Pintz in (3.3).

4. Concluding remarks

The estimation in (2.1) shows that $H(n) > c\sqrt{\frac{n}{A(n)\cdot\omega(n)\cdot\log^2 n}}$ for infinitely many n. Let us note that we could not expect more. Namely let A be the sequence of integers defined with the following properties:

(i) For every $d > 2$ and r, $0 \leqslant r < d$ there are infinitely many elements of $a \in A$ satisfying

$$a \equiv r \pmod{d},$$

(ii) choose $A(x)$ so small that say $A(n)\cdot\omega(n)\cdot\log^2 n = \log^{2+\varepsilon}$.

Now if A is thin then we can produce a Hilbert cube with dimension $\sim \frac{\sqrt{n}}{\log^{1+\varepsilon}}$, but a deep result of Szemerédi and Vu [10] shows that there is no a cube which avoids A and for which $H(n) > C\sqrt{n}$ $n > n_0$ if C is large enough.

Theorem 4.1 *(Szemerédi, Vu). There exists a $C > 0$ such that if $A \subseteq \mathbb{N}$ and*

$$A(n) > C\sqrt{n},$$

for $n > n_0$ then there exists an infinite arithmetic progression $\{z_0 + k\cdot d\colon k \in \mathbb{N}\}$ for which

$$\{z_0 + k\cdot d\colon k \in \mathbb{N}\} \subseteq FS(A).$$

Proof. Assume contrary to our assertion that the sequence A contains a cube with $H(n) > C\sqrt{n}$ $n > n_0$.

Then by Theorem 4.1 the set $FS(\{x_1, x_2, \ldots\})$ contains an infinite arithmetic progression $\{z_0 + k\cdot d\colon k \in \mathbb{N}\}$. Furthermore

$$H(a_0, x_1, x_2, \ldots) = a_0 + FS\bigl(\{x_1, x_2, \ldots\}\bigr),$$

hence

$$\{z_0 + a_0 + k \cdot d\colon\ k \in \mathbb{N}\} \subseteq H(a_0, x_1, x_2, \ldots),$$

but there is an element a in A for which $a \geqslant z_0 + a_0$ and $a \equiv z_0 + a_0 \pmod d$ which contradicts to the assumption that H avoids the sequence A. □

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