ON INTERSECTING PROPERTIES OF PARTITIONS OF INTEGERS

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ABSTRACT. We derive a generalization of a theorem of Raimi proving there is a partition of natural numbers with given densities of classes which meet structured translates of any other class of a partition of natural numbers.

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1. Introduction

A branch of combinatorial analysis – called Ramsey theory – investigates partitions of certain structures. In [1], p.180, Th 11.15] Hindman deals with the intersecting properties of a finite partition of the set $\mathbb{N}$ of positive integers. He gives an elementary proof for Raimi's theorem [2] which reads as follows:

Theorem 1.1. *There exists $E \subseteq \mathbb{N}$ such that, whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} D_i$ there exist $i \in \{1, 2, \ldots, r\}$ and $k \in \mathbb{N}$ such that $(D_i + k) \cap E$ is infinite and $(D_i + k) \setminus E$ is infinite.*

Hindman shows that the set E of natural numbers whose last non-zero entry in their ternary expansion is 1 satisfies this condition. Raimi's original proof used a topological result.

The aim of this paper is to generalize Raimi's Theorem which will be done in the next section.

2. A generalization of Raimi's Theorem

Let $\mathbb{N}$ be the set of non-negative integers. Let $A \subseteq \mathbb{N}$ and let $b \in \mathbb{N}$. Then $A + b = \{a + b : a \in A\}$. Given a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{N}$,

$$FS(\{x_n\}_{n=1}^{\infty}) = \{\textstyle\sum_{n\in F} x_n : F \text{ is a finite nonempty subset of } \mathbb{N}\}.$$

For $A \subseteq \mathbb{N}$ let us define the lower density of A by $\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1, n]|}{n}$, the upper density by $\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{n}$, and the density by $d(A) = \lim_{n\to\infty} \frac{|A \cap [1, n]|}{n}$ if the limit exists. Given a real number x we denote by $\langle x\rangle$ the fractional part of x. That is, $\langle x\rangle = x - \lfloor x\rfloor$. Given a subset A of $\mathbb{R}$, we write $\mu(A)$ for the outer Lebesgue measure of A.

Now we state a generalization of Raimi's theorem.

Theorem 2.1. *Let $A \subseteq \mathbb{N}$ such that there is a positive irrational γ for which $\{\langle\gamma x\rangle : x \in A\}$ is dense in $[0,1)$. Let $r \in \mathbb{N}$ and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive real numbers such that $\sum_{i=1}^r \alpha_i = 1$. There exists a disjoint partition $\mathbb{N} = \bigcup_{i=1}^r$ such that*
(1) for every $i \in \{1,2,\ldots,r\}$, $d(E_i) = \alpha_i$ and
(2) for each $t \in N$ and each partition $A = \bigcup_{j=1}^t F_j$, there exist $m \in \{1,2,\ldots,t\}$ and a sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{N}$ such that for every $h \in FS(\{x_n\}_{n=1}^\infty)$ and every $i \in \{1,2,\ldots,r\}$, $(F_m + h) \cap E_i$ is infinite.

Notice that Raimi's theorem follows from the case $r = 2$.

First we prove a technical lemma.

Lemma 2.2. *Let $\{I_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint intervals in $[0,1)$ and assume that for every $\varepsilon > 0$ there exist $a \in [0,1)$ and $m \in \mathbb{N}$ such that $\bigcup_{n=m+1}^\infty I_n \subseteq (a, a+\varepsilon)$. Let γ be a positive irrational number, and let $E = \{x \in \mathbb{N} : \langle\gamma x\rangle \in \bigcup_{n=1}^\infty I_n\}$. Then $d(E) = \sum_{n=1}^\infty \mu(I_n)$.*

Proof of Lemma 2.1. Recall that if γ is a nonzero irrational number, then $\{\langle\gamma x\rangle : x \in \mathbb{N}\}$ is uniformly distributed mod 1. That is, if $0 \le a < b \le 1$, then $d(\{x \in \mathbb{N} : \langle\gamma x\rangle \in (a,b)\}) = b - a$. Let $\alpha = \sum_{n=1}^\infty \mu(I_n)$. Let $\varepsilon > 0$ be given and let $k \in \mathbb{N}$ be an integer such that $\sum_{n=1}^k \mu(I_n) > \alpha - \varepsilon$. Choose an $a \in [0,1)$ and $m \in \mathbb{N}$ such that $\bigcup_{n=m+1}^\infty I_n \subseteq (a, a+\varepsilon)$. We may presume that $m \ge k$.

Let $F = \{x \in \mathbb{N} : \langle\gamma x\rangle \in \bigcup_{n=1}^m I_n\}$ and let $G = \{x \in \mathbb{N} : \langle\gamma x\rangle \in \bigcup_{n=1}^m I_n \cup (a, a+\varepsilon)\}$. Now $\bigcup_{n=1}^m I_n \cup (a, a+\varepsilon)$ is a finite union of pairwise disjoint intervals of total length $\delta \le \sum_{n=1}^m \mu(I_n) + \varepsilon$. Therefore we have by the uniform distribution of $\{\langle\gamma x\rangle : x \in \mathbb{N}\}$ that $d(F) = \sum_{n=1}^m \mu(I_n)$ and $d(G) = \delta$. Thus $\underline{d}(E) \ge d(F) \ge \sum_{n=1}^k \mu(I_n) > \alpha - \varepsilon$ and $\overline{d}(E) \le d(G) \le \sum_{n=1}^m \mu(I_n) + \varepsilon \le \alpha + \varepsilon$. □

Proof of Theorem 2.2. Take a positive irrational γ for which $\{\langle\gamma x\rangle : x \in A\}$ is dense in $[0,1)$. Let $s_0 = 0$ and inductively for $i \in \{1,2,\ldots,r\}$, let $s_i = s_{i-1} + \alpha_i$ (so $s_r = 1$). For $i \in \{1,2,\ldots,r\}$ and $j \in \mathbb{N}$, let

$$J_{i,j} = \left[1 - \frac{1}{2^j} + \frac{s_{i-1}}{2^{j+1}}, 1 - \frac{1}{2^j} + \frac{s_i}{2^{j+1}}\right).$$

For $i \in \{1,2,\ldots,r\}$ let $J_i = \bigcup_{j=0}^\infty J_{i,j}$ and let $E_i = \{x \in \mathbb{N} : \langle\gamma x\rangle \in J_i\}$. Then $\mu(J_i) = \sum_{j=0}^\infty \dfrac{s_i - s_{i-1}}{2^{j+1}} = \alpha_i$ so by the lemma, $d(E_i) = \alpha_i$.

Now let $t \in N$ and let $A = \bigcup_{j=1}^t F_j$. We claim

Fact: *For any c, d with $0 \le c < d \le 1$ there exists $m \in \{1, 2, \ldots, t\}$ and there exist a, b, with $c \le a < b \le d$ such that $\{\langle \gamma x \rangle : x \in F_m\}$ is dense in (a, b).*

To see this, suppose not. Let $a_0 = c$ and $b_0 = d$. Inductively let $j \in \{1, 2, \ldots, t\}$. Then $\{\langle \gamma x \rangle : x \in F_j\}$ is not dense in (a_{j-1}, b_{j-1}) so pick a_j, b_j with $a_{j-1} \le a_j < b_j \le b_{j-1}$ such that $\{\langle \gamma x \rangle : x \in F_j\} \cap (a_j, b_j) = \emptyset$. When this process is complete one has that $(a_t, b_t) \cap \bigcup_{j=1}^t \{\langle \gamma x \rangle : x \in F_j\} = \emptyset$. That is, $(a_t, b_t) \cap \{\langle \gamma x \rangle : x \in A\} = \emptyset$, a contradiction.

Now for $n \in \mathbb{N}$, we inductively choose a_n, b_n, and $m(n)$ such that $m(n) \in \{1, 2, \ldots, t\}$, $0 < a_n < b_n < 1$, $\{\langle \gamma x \rangle : x \in F_{m(n)}\}$ is dense in (a_n, b_n), $b_n \le a_{n+1}$, $a_{n+1} \ge 1 - \dfrac{b_n - a_n}{4}$, and $b_{n+1} - a_{n+1} \le \dfrac{b_n - a_n}{2}$.

Choose $m(1) \in \{1, 2, \ldots, t\}$ and a_1, b_1 such that $0 < a_1 < b_1 < 1$ and $\{\langle \gamma x \rangle : x \in F_{m(1)}\}$ is dense in (a_1, b_1). Given $n \in \mathbb{N}$ and a_n and b_n, let $c = \max\left\{b_n, 1 - \dfrac{b_n - a_n}{4}\right\}$ and $d = \min\left\{1, c + \dfrac{b_n - a_n}{2}\right\}$. Apply Fact to choose $m(n+1) \in \{1, 2, \ldots, t\}$ and a_{n+1}, b_{n+1} with $c \le a_{n+1} < b_{n+1} \le d$ such that $\{\langle \gamma x \rangle : x \in F_{m(n+1)}\}$ is dense in (a_{n+1}, b_{n+1}). Then $b_n \le c \le a_{n+1}$, $1 - \dfrac{b_n - a_n}{4} \le c \le a_{n+1}$, and $b_{n+1} \le d \le c + \dfrac{b_n - a_n}{2} \le a_{n+1} + \dfrac{b_n - a_n}{2}$.

Now take $m \in \{1, 2, \ldots, t\}$ such that $D = \{n : m(n) = m\}$ is infinite and enumerate D in increasing over as $\{n(k)\}_{k=1}^\infty$. For each $k \in \mathbb{N}$, let $c_k = a_{n(k)}$ and $d_k = b_{n(k)}$. Then for each k, $\{\langle \gamma x \rangle : x \in F_m\}$ is dense in (c_k, d_k), $d_k \le c_{k+1}$, $c_{k+1} \ge 1 - \dfrac{d_k - c_k}{4}$, and $d_{k+1} - c_{k+1} \le \dfrac{d_k - c_k}{2}$. For each $k \in \mathbb{N}$ pick $x_k \in \mathbb{N}$ such that $\langle \gamma x_k \rangle \in \left(1 - d_k, 1 - c_k - \dfrac{d_k - c_k}{2}\right)$.

Notice that for any $k \in \mathbb{N}$ and $v \in \omega$, $d_{k+v} - c_{k+v} \le \dfrac{d_k - c_k}{2^v}$.

We show now by induction on $v \in \mathbb{N}$ that

$$H \subseteq \mathbb{N}\,,\ |H| = v, \text{ and } k = \min H \Rightarrow$$

$$\Rightarrow \langle \gamma \textstyle\sum_{l \in H} x_l \rangle \in \left(1 - d_k, 1 - c_k - \dfrac{d_k - c_k}{2^v}\right). \qquad (**)$$

When $v = 1$, $(**)$ holds directly, so assume that $v > 1$ and $(**)$ holds for $v - 1$. Let $H \subseteq \mathbb{N}$ with $|H| = v$, let $k = \min H$, let $u = \max H$, and let $G = H \setminus \{u\}$. Then $\langle \gamma \sum_{l \in G} x_l \rangle < 1 - c_k - \dfrac{d_k - c_k}{2^{v-1}}$ and $\langle \gamma x_u \rangle < 1 - c_u \le 1 - c_{k+v-1} \le \dfrac{d_{k+v-2} - c_{k+v-2}}{4} \le \dfrac{d_k - c_k}{2^v}$ so

$\langle\gamma\sum_{l\in G}x_l\rangle+\langle\gamma x_u\rangle<1-c_k-\dfrac{d_k-c_k}{2^{v-1}}+\dfrac{d_k-c_k}{2^v}=1-c_k-\dfrac{d_k-c_k}{2^v}$. Since $\langle\gamma\sum_{l\in G}x_l\rangle+\langle\gamma x_u\rangle<1$, we have that $\langle\gamma\sum_{l\in G}x_l\rangle+\langle\gamma x_u\rangle=\langle\gamma\sum_{l\in H}x_l\rangle$ and so $(**)$ is established.

Now let H be a finite nonempty subset of $\mathbb{N}$ and let $i\in\{1,2,\dots,r\}$. We show that $(F_m+\sum_{l\in H}x_l)\cap E_i$ is infinite. Let $k=\min H$. Then by $(**)$, $\langle\gamma\sum_{l\in H}x_l\rangle\in(1-d_k,1-c_k)$ so $c_k+\langle\gamma\sum_{l\in H}x_l\rangle<1<d_k+\langle\gamma\sum_{l\in H}x_l\rangle$. Pick $j\in\mathbb{N}$ such that $1-\dfrac{1}{2^j}>c_k+\langle\gamma\sum\limits_{l\in H}x_l\rangle$. Then

$c_k<1-\dfrac{1}{2^j}-\langle\gamma\sum\limits_{l\in H}x_l\rangle+\dfrac{s_{i-1}}{2^{j+1}}<1-\dfrac{1}{2^j}-\langle\gamma\sum\limits_{l\in H}x_l\rangle+\dfrac{s_i}{2^{j+1}}<d_k$ and $\{\langle\gamma y\rangle:y\in F_m\}$ is dense in (c_k,d_k) and so

$$K=\left\{y\in F_m:1-\frac{1}{2^j}-\langle\gamma\sum_{l\in H}x_l\rangle+\frac{s_{i-1}}{2^{j+1}}<\langle\gamma y\rangle<1-\frac{1}{2^j}-\langle\gamma\sum_{l\in H}x_l\rangle+\frac{s_i}{2^{j+1}}\right\}$$

is infinite.

To complete the proof it suffices to show that if $y\in K$, then $y+\sum_{l\in H}x_l\in E_i$. Indeed, given $y\in K$, $\langle\gamma y\rangle+\langle\gamma\sum_{l\in H}x_l\rangle\in J_{i,j}$ and $\langle\gamma y\rangle+\langle\gamma\sum_{l\in H}x_l\rangle<1$ so $\langle\gamma y\rangle+\langle\gamma\sum_{l\in H}x_l\rangle=\langle\gamma(y+\sum_{l\in H}x_l)\rangle$ so $y+\sum_{l\in H}x_l\in E_i$ as required.

□

3. Further problems and results

Theorem 2.2 implies that for every t partition of the set $\mathbb{N}=\bigcup_{j=1}^t F_j$ not just one translation h of some F_m meets E_j : $(j=1,\dots,r)$ in an infinite set, rather each translations do, given h from an additive "cube".

A natural question is to ask the following: Is any infinite set $\{x_n\}_{n=1}^\infty$, such that Theorem 2.2 remains true if we want that the elements h included in $FS(\{x_n\}_{n=1}^\infty)\cup FP(\{x_n\}_{n=1}^\infty)$, where $FP(\{x_n\}_{n=1}^\infty)$ is a multiplicative cube defined by

$FS(\{x_n\}_{n=1}^\infty)=\{\prod_{n\in F}x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$?

Our combinatorial approach is not enough to prove this extension. Maybe some tools from ergodic theory would work.

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