IP sets, Hilbert cubes

by

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Abstract: Given subset E of natural numbers FS(E) is defined as the collection of all sums of elements of finite subsets of E and any translation of FS(E) is said to be Hilbert cube. We estimate a growing of E for which FS(E) avoids a set of multiplies of a given infinite set. The results are related to a result which states that there exists an infinite Hibert cube containing in set of square-free numbers.

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1. INTRODUCTION

There is an interesting and permanent dialog between ergodic theory and combinatorial number theory. An important configuration in ergodic theory and in the combinatorial number theory as well, sets called $IP-set$ and Hilbert cube. Given subset E of natural numbers $FS(E)$ is defined as the collection of all sums of elements of finite subsets of E (sometimes the empty set is excluded). If E is infinite then the set $FS(E)$ is said to be $IP-set$ (which is an important notion in ergodic theory). A more general configuration is a Hilbert cube which is a translation of $FS(E)$, i.e. if $E = \{x_1, x_2, \ldots\}$ is a finite or infinite set of integers, a_0 is a given integer, then

$$H = H(a_0, x_1, x_2, \ldots) = \{a_0 + \sum \varepsilon_i x_i : \varepsilon_i \in \{0, 1\}; \sum \varepsilon_i < \infty\}$$

called Hilbert-cube (briefly H-cube) or combinatorial cube. Throughout of the paper we consider those cubes where the vertices are different, i.e. H-cubes in the form $H = H(a_0, x_1 < x_2 < \ldots)$.

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The existence of "big" H-cube in dense set was a crucial point of Szemerédi's proof of his theorem on arithmetic progressions,. Many author investigated infinite H-cubes in special sets (see e.g. [HI], [HE-SÁ], [BE-RU].)

A stronger version of an $IP-set$ will be $\overline{IP}$-set. We define $\overline{FS(A)}$ the set of all sums of the form $\sum x_i a_i$ where x_i is an integer and fulfils the condition $0 \leq x_i \leq i$, furthermore we assume that $\sum x_i < \infty$. If a set contains an $\overline{FS(A)}$ then it is said to be $\overline{IP}$-set.

It is reasonable to distinguish the notion of IP-set ($\overline{IP}$-set) and Hilbert cube. Clearly the set of odd numbers contains an infinite Hilbert cube, but does not contain an IP-set (in fact an $FS(E)$ is a subset of the set of odd numbers if and only if E contains just only one odd number, i.e. $E = \{2k-1\}$ for some natural number k). An interesting result of E.G. Strauss is that for every $\varepsilon > 0$ there exists a sequence A for which $d(A) = 1 - \varepsilon$ nevertheless A does not contain an infinite Hilbert cube (see in [HI]). Therefore it is a harder job to show that a set does not contain an infinite Hilbert cube or to prove that a given set contains an IP-set.

In [BE-RU] Bergelson and Ruzsa proved the following interesting fact: Let A be the sequence of squarefree numbers. The set $A - a$ contains an $IP - set$ if and only if $a \in A$ (i.e. for every $a \in A$ there exists an infinite Hilbert cube $H = H(a, x_1, x_2, \ldots)$ in A).

They derived this result from the following Theorem:

THEOREM A:

Let $X \subseteq N$ be a set such that $1 \notin X$, any two elements of X are coprime and

$$\sum_{x \in X} 1/x < \infty. \tag{1}$$

Define $B^c(X)$ as the set of natural numbers that are not divisible by any element of X. Then $B^c(X)$ contains an $IP - set$.

The aim of this note is to investigate related questions. We can ask a generalization of this result as follows:

Let S be an arbitrary semigroup, and let A be a subset of it. Assume that there exists an n, such that any j-element subset of A, $1 \leq j \leq n$ does not generate S. Let $B(A) = \bigcup_i < A_i >$, where for every i the cardinality of A_i is at most n, and $\bigcup_i A_i$ is a (disjoint) partition of A. Let $B^c(A) = S \setminus B(A)$. Our question is the following: for which A will contain $B^c(A)$ an infinite IP-set (Hilbert-cube)? If $S = N$, and A_i's are the one-element subsets of A then we obtain the original question of Bergelson and Ruzsa. In the present paper we are going to investigate the case when $S = N^k$.

2. NOTATIONS:

For $A \subseteq \mathbf{N}^k$ we define $B(A)$ as the set of all multiple of elements of A, i.e., $B(A) = \bigcup_{a \in A} < a > \cap \mathbf{N}^k = \{\lambda a : a \in A \ \lambda \in \mathbf{N}\}$. Let $B^c(A) = N^k \setminus B(A)$ (the set of all vectors in $\mathbf{N}^k$ which are not a multiple of any elements of A). $\mathbf{N}$ denotes the set of non-negative integers, $\mathbf{R}$ the set of real numbers. Let us denote by $d(P,Q)$ the Euclidean distance between $P, Q \in \mathbf{R}^k$. If the point Q is the origin, then we write $d(P)$. The counting function $A(x)$ of $A \subseteq \mathbf{N}^k$ is defined by $A(x) = \sum_{a \in A; d(a) \le x} 1$. $|\cdot|$ denotes the cardinality of a set. Let $A, B \subseteq \mathbf{R}^k$. The set addition of A and B is defined by $A + B = \{a + b | a \in A, b \in B\}$. If $C \subseteq \mathbf{R}^k$ then let $\lambda C = \{\lambda c : c \in C\}$.

3. A result in $\mathbf{N}^k$

In the present section we prove a related result of Bergelson-Ruzsa's result in $k-$dimension:

THEOREM 1:

Let $A \subseteq N^k$, $k \ge 2$, and assume that

$$\liminf_{x \to \infty} \frac{A(x)}{x^{k-1}} = 0. \tag{2}$$

Then $B^c(A)$ contains an $\overline{IP}$-set.

PROOF OF THEOREM 1:

Our task is to find an $\overline{IP}$-set containing in $B^c(A)$. By (2) we have that $B^c(A)$ is a non-empty set, therefore we can select an $\vec{x_1} \in B^c(A)$. Furthermore let us assume that the set $\overline{FS(X)}$ has been defined and we have $\overline{FS(X)} = \{\alpha_1 \vec{x_1} + \alpha_2 \vec{x_2} + \ldots + \alpha_n \vec{x_n} : \alpha_i \in \mathbf{N};\ 0 \le \alpha_i \le i\} \subseteq B^c(A)$.

Let $M = M(n) = \sum_{1 \le i \le n} i \cdot d(\vec{x_i})$. Consider a k-dimensional ball

$$G_k(M) = \{\vec{x} : \vec{x} \in \mathbf{R}^k;\ d(\vec{x}) \le M\},$$

By the definition of M we conclude that $\overline{FS(X)}$ is a subset of $G_k(M) \cap \mathbf{N}^k$. By (2) we have that there exists an $x > 2n \cdot M$ for which

$$A(x) \le c_1 \big(\frac{x}{M}\big)^{k-1}$$

holds, where c_1 will be specified later. Let S be a $k-1$ dimensional surface of a sphere with radius x, i.e. $S = \{\vec{z} : \vec{z} \in \mathbf{R}^k; d(\vec{z}) = x\}$. Set a dense packing of S by $(k-1)$-dimensional balls with radius $2nM$, where packing means that any two different balls have at most one common point, and dense means that the volume of the union of the balls is a positive proportion of the volume of S, i.e. if $G_1, G_2, \dots, G_s$ are the balls then there is a constant $c(k) = c > 0$ such that

$$c \cdot vol(S) < vol\big(\cup_{i=1}^{s} G_i\big) = \sum_{i=1} vol(G_i) = s \cdot (c_2 M^{k-1}),$$

where c_2 depends only on k. Hence we obtain

$$s > \frac{c \cdot vol(S)}{c_2 M^{k-1}} = c_3 \big(\frac{x}{M}\big)^{k-1},$$

where c_3 depends only on k.

The convex hull of the origin and a G_i is said to be *cell* and denoted by C_i. The number of cells is at least

$$c_3 \big(\frac{x}{M}\big)^{k-1}.$$

Since $A(x) \le c_1 \big(\frac{x}{M}\big)^{k-1}$, if $c_1 < c_3$ then there is a cell C which does not contain an element from A. We claim that the inside of C and $B(A)$ are disjoint sets. Assume now contrary to the assumption that there is a $\vec{b} \in B(A)$ and $\vec{b}$ lies in the inside of C. Since $\vec{b} \in B(A)$ we have that there is a positive integer t such that $\vec{b}/t \in A$. But C is a convex set thus $\vec{b}/t$ lies in C which contradicts to the fact that C and A are disjoint sets.

Dilate now C into $\frac{1}{n+1}C$. By the definition of C we infer that $\frac{1}{n+1}C$ contains a k-dimensional ball G^* with radius M and center from $\mathbf{N}^k$. Let x_{n+1} be the center of G^*.

Recall that

$$\overline{FS(X)} = \{\alpha_1 \vec{x_1} + \alpha_2 \vec{x_2} + \ldots + \alpha_n \vec{x_n} : \alpha_i \in \mathbf{N};\ 0 \leq \alpha_i \leq i\}$$

and clearly

$$\overline{FS(X \cup \{x_{n+1}\})} = \bigcup_{i=1}^{n+1} (\overline{FS(X)} + i \cdot x_{n+1}).$$

Consider the sequence of balls

$$G^* = G_1^*; G_2^*; \ldots ; G_{n+1}^*,$$

where the radius of G_i^* is M and the center is $i \cdot x_{n+1}$. Note that for all $G_i^* \subseteq C$. Furthermore observe that for each i;1$\leq i \leq n+1$

$$i \cdot x_{n+1} + \overline{FS(X)} \subseteq G_i^*,$$

and hence the set of union of these sets also lies in C. It means that

$$\bigcup_{i=1}^{n+1} (\overline{FS(X)} + i \cdot x_{n+1}) \cap B(A) = \emptyset,$$

i.e.

$$\overline{FS(X \cup \{x_{n+1}\})} \cap B(A) = \emptyset,$$

as we wanted.

Hence the theorem.

4. A result in N

In the present paragraph we shall investigate how sharp is the Bergelson-Ruzsa's theorem. What happens in Theorem A if we replace a convergent series to a divergent one; what can we say about $B^c(A)$? Trivially for the sequence of primes, which forms a divergent series, the set $B^c(A)$ will not contain an infinite Hilbert cube since it will be the empty set. But on the other hand it is not completely clear that one can leave *infinitely* many primes from P of having P', and $B^c(P')$ does not contain an Hilbert cube. First we shall prove:

PROPOSITION:

There exists a subset P' of the set of primes P for which $|P \setminus P'| = \infty$, and $B^c(P')$ does not contain an infinite Hilbert cube.

Furthermore we give an estimation that how "big" could be a Hilbert cube containing in $B^c(A)$, where A is a subsequence of all primes. So let $H = H(a_0, x_1, x_2, \ldots)$ be a Hilbert cube in $B^c(A)$ and let

$$H_A(n) = H(n) = |\{x_1 < x_2 < \ldots\} \cap [1, n]|,$$

i.e. $H(n)$ counts the number of vertices of a H-cube up to n.

We prove

THEOREM 2:

Let $A = \{p_1 < p_2 < \ldots < p_n < \ldots\}$ be an infinite sequence of primes. Then

$$H(n) < 8 \sum_{i=1}^{f(n)} p_i^{3/2}, \tag{3}$$

where $f(n)$ is the smallest s, for which $p_1 p_2 \ldots p_s \geq n$.

We apply Theorem 1 in the following case:

COROLLARY:

Let $\alpha > 1$ and let $P' = \{p_1 < p_2 < \ldots <\}$ be a sequence of primes for which $\lim_{i \to \infty} \frac{p_i}{i^\alpha} = 1$. Then we have

$$H_{P'}(n) < c(\alpha) \Big(\frac{logn}{loglogn} \Big)^{\frac{3\alpha+2}{2}}.$$

PROOF OF THE COROLLARY:

First let us note that for every $\alpha > 1$ there is a sequence P' of primes for which $\lim_{i \to \infty} \frac{p_i}{i^\alpha} = 1$. Indeed from the Prime Number Theorem we have that $q_i \sim i \cdot logi$, where q_i is the i^{th} prime of the sequence of all primes. Now selecting an arbitrary sequence $i_j \sim \frac{j^\alpha}{\alpha \log j}$, we obtain

$$p_j = q_{i_j} \sim i_j \cdot logi_j \sim j^\alpha.$$

Now using the Stirling's formula we get that $f(n) = c_1 \frac{\log n}{\log \log n}$, where c_1 depends only on α. By Theorem 2 we have

$$H_{P'}(n) < 8 \sum_{i=1}^{f(n)} p_i^{3/2} < c_1 \int_{i=1}^{f(n)} i^{3\alpha/2} <$$

$$< c(\alpha) \Big(\frac{logn}{loglogn} \Big)^{\frac{3\alpha+2}{2}}.$$

REMARK:

The related question of Bergelson and Ruzsa is the following: for a given sequence A of primes $B(A) = \{k \cdot p : \ p \in A; \ k \in \mathbf{N}\}$ and the complement of $B(A)$ is the set of all integers which composed solely of the primes are not in A. In the Proposition we show the existence an infinite cube in $B^c(A)$.

We can ask an opposite question as well: Let A' be any sequence of prime numbers and let $\prod(A')$ be the set of all integers can be divided by primes only from A. Observe that $\prod(A')$ is a *generalized* multiplicative $\overline{IP}$-set. Then we ask: what is the maximal additive Hilbert cube which avoids $\prod(A')$? We return to this question in an other paper.

PROOF OF PROPOSITION 1:

For the proof we need a lemma which is a special case of a deep result of R. Tijdeman (see in [T]).

Define by $U(P")$ the set of all integers which composed solely of the primes of $P"$, i.e. $U(P") = \{n : p|n \Rightarrow p \in P"\} = \{n_1 < n_2 < \ldots < n_i < \ldots\}$.

LEMMA 1:

There exists an infinite sequence $P"$ of primes for which

$$n_{i+1} - n_i > \sqrt{n_i}. \tag{4}$$

Note that Lemma 1 also implies that

$$n_i - n_{i-1} > \frac{\sqrt{n_i}}{2}. \tag{5}$$

Indeed if $n_i \le 4n_{i-1}$ then $n_i - n_{i-1} > \sqrt{n_{i-1}} \ge \frac{\sqrt{n_i}}{2}$ or $n_i > 4n_{i-1}$ and then $n_i - n_{i-1} > \frac{3}{4}n_i > \frac{\sqrt{n_i}}{2}$.

Now let $P' = P \setminus P"$. By Lemma 1 we get that $|P \setminus P'| = |P"| = \infty$. By the definition of $B(P')$, an integer m is an element of $B(P')$ if m has a prime divisor from the set P'. Hence $B^c(P')$ collects all integers composed solely of the primes $P"$, i.e. $m \in B^c(P')$ if and only if $m = \prod_{p \in P"} p^\alpha$ and hence $B^c(P') = U$.

Now assume contrary to the assertion that $B^c(P')$ contains an infinite Hilbert cube, say $H = H(a_0, x_1 < x_2 < \ldots)$. Choose an element h from H, for which $h > 4x_1^2$. Since H is infinite such an element exists. By (4) and (5) we obtain

$$min\{|m - h| : m \in B^c(P')\} > \frac{\sqrt{h}}{2} > x_1. \tag{6}$$

But h can be expressed as $h = h' + \varepsilon_1 x_1$, where $\varepsilon_1 = 0$ or $\varepsilon_1 = 1$, and x_1 is not a term of h'. If $\varepsilon_1 = 0$ then $h + x_1 \in H$, if $\varepsilon_1 = 1$ then $h - x_1 \in H$. Both cases contradicts to (6).

PROOF OF THEOREM 1:

Assume that there exists an infinite cube $H = H(a_0, x_1 < x_2 < \ldots) \subseteq B^C(A)$. We shall prove

$$H(n) < 4 \sum_{i=1}^{f(n)} p_i^{3/2}.$$

where $A = \{p_1 < p_2 < \ldots < p_n < \ldots\}$. Now for every $k \le f(n)$ write $p_k = Mp + r,\ 0 < r < p$, where p is an arbitrary prime less than p_k. Clearly there exists a positive integer r' such that

$$r \cdot r' = p \cdot s + a_0,$$

for some $s \in \mathbf{N}$. Since $p_k \in A$ we obtain that for every $w \in \mathbf{N}$, $(pw + r')p_k \in B(A)$ and hence

$$(pw + r')p_k = p(p_k w + r'M + s) + a_0.$$

Define H_k the set of all elements of $\{x_1 < x_2 < \ldots\} \cap [1, n]$ which are not divisible by p_k. Thus if

$$x_u \notin \bigcup_{k=1}^{f(n)} H_k$$

then we conclude

$$p_1 p_2 \cdots p_k | x_u \ \Rightarrow\ x_u > n,$$

i.e. $\{x_1 < x_2 < \ldots\} \setminus \cup_{k=1}^{f(n)} H_k$ are bigger than n. Hence we have

$$H(n) \leq \sum_{k=1}^{f(n)} |H_k|.$$

We give an upper estimation for H_k. Let $y_1 < y_2 < \ldots y_u \leq n$ be the elements of H_k. We shall prove $u \leq 4p_k^{3/2}p$. Assume contrary to this assertion that $u > 4p_k^{3/2}p$. Split the set $\{y_1 < y_2 < \ldots < y_u\}$ into $2p$ many pairwise disjoint sets with cardinality at least $2p_k^{3/2}$. Denote they by $Y_1, Y_2, \ldots Y_m, \ldots Y_{2p}$. We claim that it is enough to prove that for each set Y_m there exists an n_m such that

$$p_k n_m + r'M + s \in FS(Y_m). \tag{5}$$

Indeed by the Erdős-Ginzburg-Ziv theorem (see [EGZ]) we can select p elements from the set $\{n_m\}_{1 \leq m \leq 2p}$ having a sum $\equiv 0 (mod\ p)$ i.e.

$$\sum_{j=1}^{p} (p_k n_{m_j} + r'M + s) = p(p_k \cdot w + r'M + s),$$

with some integer w. But $(pw + r')p_k = p(p_k w + r'M + s) + a_0 \in B(A) \cap H(a_0, x_1, x_2 \ldots)$ a contradiction.

Let $Y_m = Y = \{y_1, y_2, \ldots, y_s\}$, $s \geq 2p^{3/2}$. We distinguish two cases; If there is an $r \not\equiv 0\ (modp_k)$ such that $y_i \equiv r\ (modp_k)$ has at least p_k solution, then for some z, $z \cdot r \equiv r'M + s\ (modp_k)$, which implies (5).

If for any $r \not\equiv 0\ (modp_k)$ $y_i \equiv r\ (modp_k)$ has at most $p_k - 1$ solution, then we can select $Y' = \{y_{i_1}, \ldots, y_{i_T}\} \subseteq Y$, for which the elements of Y' are pairwise incongruent and $T > 2\sqrt{p_k}$.

LEMMA 2:

Assume $Y' = \{y_{i_1}, \ldots, y_{i_T}\}$ a set of integers for which the elements of Y' are pairwise incongruent and $T > 2\sqrt{p_k}$. Then $\sum(Y')$ intersects every residue classes $(mod\ p_k)$.

This is a theorem of Olson $[OL]$.

But it gives again (5).

Finally note that if $X \subseteq X'$ then $B^c(X') \subseteq B^c(X')$, hence without loss of generality $2 < p_1$..

We obtained $H_k \leq 4p_k^{3/2}p$, and since this argument holds for all prime p for the smallest we get

$$H_k \leq 8p_k^{3/2},$$

which proves the theorem.

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