A NOTE ON FREIMAN MODELS IN HEISENBERG GROUPS

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ABSTRACT. Green and Ruzsa recently proved that for any $s \geq 2$, any small squaring set A in a (multiplicative) abelian group, i.e. $|A \cdot A| < K|A|$, has a Freiman s-model: it means that there exists a group G and a Freiman s-isomorphism from A into G such that $|G| < f(s, K)|A|$.

In an unpublished note, Green proved that such a result does not necessarily hold in non abelian groups if $s \geq 64$. The aim of this paper is improve Green's result by showing that it remains true under the weaker assumption $s \geq 6$.

1. **Introduction**

We will use the notation $|X|$ for the cardinality of any set or group X. If X and Y are subsets of a given (multiplicative) group, the product $X \cdot Y$ or simply XY denotes the set $\{xy \mid x \in X, y \in Y\}$. For $X = Y$ we write $XY = X^2$. The set X^{-1} is formed by all the inverse elements x^{-1}, $x \in X$.

Let $s \geq 2$ be an integer and $A \subset H$ and $B \subset G$ be subsets of arbitrary (multiplicative) groups. A map $\pi : A \to B$ is said to be a Freiman s-homomorphism if for any $2s$-tuple $(a_1, \ldots, a_s, b_1, \ldots, b_s)$ of elements of A and any signs $\epsilon_i = \pm 1$, $i = 1, \ldots, s$, we have

$$a_1^{\epsilon_1} \ldots a_s^{\epsilon_s} = b_1^{\epsilon_1} \ldots b_s^{\epsilon_s} \Longrightarrow \pi(a_1)^{\epsilon_1} \ldots \pi(a_s)^{\epsilon_s} = \pi(b_1)^{\epsilon_1} \ldots \pi(b_s)^{\epsilon_s}.$$

Observe that in the case of abelian groups, we may set, without loss of generality, all the signs to $+1$. If moreover π is bijective and π^{-1} is also a Freiman s-homomorphism, then π is called a Freiman s-isomorphism from A into G. In this case, A and B are said to be Freiman s-isomorphic.

Green and Ruzsa proved in [2] that a structural result holds for small squaring sets in an abelian (multiplicative) group. The key argument in their proof is Proposition 1.2 of [2] asserting that any small squaring finite set A in an abelian group has a good Freiman model, that is a relatively small finite group G and a Freiman s-isomorphism from A into G. More precisely, they showed the following effective result:

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Let $s \geq 2$ and $K > 1$. There exists a constant $f(s,K) = (10sK)^{10K^2}$ such that A is a subset of an abelian group H satisfying the small squaring property $|A \cdot A| < K|A|$, then there exists an abelian group G such that $|G| < f(s,K)|A|$ and A is Freiman s-isomorphic to a subset of G.

It is not difficult to see that this result cannot be literally extended to nonabelian groups by considering a set A such that $|A \cdot A|/|A|$ is small and $|A \cdot A \cdot A|/|A|$ is large (see [6, page 94] for such an example). However it is known (by combining [4, section 1.11] and [6, Proposition 2.40]) that if $|A \cdot A|/|A| \leq K$ then for any n-tuple of signs $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$, we have $|X^{\epsilon_1} \cdot X^{\epsilon_2} \cdots X^{\epsilon_n}|/|X| \leq K^{O(n)}$ for some large subset X of A satisfying $|X| \geq |A|/2$. Despite this fact, the existenceness of a good Freiman s-model for some large subset of an arbitrary set A_0 satisfying the small squaring property $|A_0 \cdot A_0| < 2|A_0|$ is not guaranteed. Indeed in his unpublished note [3], Green gave an example of such a set A_0 with arbitrarily large cardinality and the following property: let $s \geq 64$ and $\delta = 1/23$; then for any $A \subset A_0$ with $|A| \geq |A_0|^{1-\delta}$ and any finite group G such that there is a Freiman s-isomorphism from A into G, we have $|G| \geq |A|^{1+\delta}$. There is no doubt from his proof that the admissible range for s could be somewhat improved ($s \geq 32$ is seemingly the best range that can be read from his proof).

Our aim is to improve Green's result by showing:

Theorem 1. *Let n be any positive integer and ε be any positive real number. Then there exists a finite (nonabelian) group H and a subset A_0 in H with the following properties:*

i) $|A_0| > n$ *and* $|A_0 \cdot A_0| < 2|A_0|$*;*
ii) *For any $A \subset A_0$ with $|A| \geq |A_0|^{43/44}$ and for any finite group G such that there exists a Freiman 6-isomorphism from A onto G, we have $|G| \geq |A|^{33/32-\varepsilon}$.*

Our proof in Section 4 is partially based on Green's approach but also includes new materials. It exploits arguments coming from group theory and Fourier analysis with additional tools, e.g. a recent incidence theorem due to Vinh [7]. It also needs some additional combinatorial arguments.

In Section 3, we include for comparison the proof of a weaker statement that does not use the new materials, but which optimizes, in some sense, Green's ideas.

Let p be a prime number and $\mathbb{F}$ the fields with p elements. We denote by H the Heisenberg linear group over $\mathbb{F}$ consisting of the upper triangular matrices

$$[x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{F}.$$

We recall the product rule in H:

$$[x, y, z] \cdot [x', y', z'] = [x + x', y + y', xy' + z + z'].$$

As shown in [3], this group provides an example of a nonabelian group in which there exists some subset A_0 with small *squaring* property, namely $|A_0^2| < 2|A_0|$, and not having a good Freiman model. That is there is no *relatively big* isomorphic image of A_0 by a Freiman s-isomorphism with a given s in any group G. We will also use the Heisenberg group in order to derive our results.

The proof of Theorem 1 goes in the following manner. We will show that: firstly there exists a non trivial p-subgroup in the subgroup generated by $\pi(A)$ in G; secondly any element in $\pi^{-1}(G)$ is the product of at most 6 elements from A or A^{-1}. The rest of the proof is based on some group-theoretical properties which are mainly taken from [3].

As indicated in [3], there is no hope to obtain an optimal result by this approach, namely a similar result with $s_0 = 2$.

2. Some properties of finite nilpotent groups and of the Heisenberg group H

For any group G, we denote by 1_G the identity element of G. Thus $[0, 0, 0] = 1_H$.

We will use the following partially classical properties:

1. H is a two-step nilpotent group (or nilpotent of class two). Indeed, the commutator of $a_1 = [x_1, y_1, z_1] \in H$ and $a_2 = [x_2, y_2, z_2] \in H$ denoted by $[a_1; a_2]$ is equal to

$$[a_1; a_2] = a_1 a_2 a_1^{-1} a_2^{-1} = [0, 0, x_1 y_2 - x_2 y_1].$$

For any $a_3 = [x_3, y_3, z_3] \in H$, we obtain

$$[[a_1; a_2]; a_3] = [0, 0, 0] = 1_H,$$

for the double commutator. Hence the result.

2. Any finite nilpotent group is the direct product of its Sylow subgroups (see 6.4.14 of [5]).

3. Any finite p-group of order p or p^2 is abelian (see 6.3.5 of [5]).

4. Assume that $A \subset H$ and π is a Freiman s-homomorphism from A into G with $s \geq 5$. We denote by $\langle \pi(A) \rangle$ the subgroup generated by $\pi(A)$. Then $\langle \pi(A) \rangle$ is a two-step nilpotent group. Indeed, for any $a, b, c \in A$, one has

$$aba^{-1}b^{-1}c = caba^{-1}b^{-1}$$

since H is a nilpotent group of class *two*. Hence

$$\pi(a)\pi(b)\pi(a)^{-1}\pi(b)^{-1}\pi(c) = \pi(c)\pi(a)\pi(b)\pi(a)^{-1}\pi(b)^{-1}$$

since π is a Freiman s-homomorphism with $s \geq 5$. It thus follows that double commutators satisfy $[[a_1; b_1]; c_1] = 1_G$ for any $a_1, b_1, c_1 \in \pi(A)$. In [3], the author observed from a direct argument that it remains true for any $a_1, b_1, c_1 \in \langle \pi(A) \rangle$: since $\langle \pi(A) \rangle$ is finite, the result will follow from the next lemma (cf. [3]).

Lemma 2. *Let* Γ *be any group and* X *a maximal subset of* Γ *such that*

(1) $$[[a; b]; c] = 1_\Gamma, \quad \text{for any } a, b, c \in X.$$

Then X *in closed under multiplication.*

For the the sake of completeness we include the proof which is in the same way as in [3].

Proof. By (1) and the following identity

(2) $$[xy; z] = [x; [y; z]] \cdot [y; z] \cdot [x; z], \quad x, y, z \in \Gamma,$$

we obtain for any $a, b, c, d \in X$, $[[ab; c]; d] = [[b; c] \cdot [a; c]; d]$. Applying again (2) with $x = [b; c]$, $y = [a; c]$ and $z = c$, yields in view of (1),

(3) $$[[ab; c]; d] = 1_\Gamma, \quad \text{for any } a, b, c, d \in X.$$

By a further application of (2) with $x = a$, $y = b$ and $z = [ab; c]$, we get by (3) $[ab; [ab; c]] = 1_\Gamma$ for any $a, b, c \in X$. By the maximal property of X, we obtain $ab \in X$ for any $a, b \in X$. □

3. Approach of the proof with a slightly weaker result

Before proving our main result, we explain the principle of the approach by showing the following weaker result in which only Freiman s-isomorphisms with $s \geq 7$ are considered.

Theorem 3. *Let n be a positive integer and θ be a real number such that*

$$\frac{11}{12} \leq \theta \leq 1$$

and let

$$\varphi_\theta = \frac{12\theta - 9}{2}.$$

Then there exists a finite group H and a subset A_0 in H satisfying the following properties:

i) $|A_0| > n$ *and* $|A_0 \cdot A_0| < 2|A_0|$*;*

ii) *For any $A \subset A_0$ with $|A| \geq |A_0|^\theta$ and for any finite group G such that there exists a Freiman 7-isomorphism from A onto G, we have $|G| \geq |A|^{\varphi_\theta}$.*

For $\theta = 13/14$, it yields the following corollary which can be compared to Theorem 1:

Corollary 4. *Let n be any positive integer. Then there exists a finite group H and a subset A_0 in H satisfying the following properties:*

i) $|A_0| > n$ *and* $|A_0 \cdot A_0| < 2|A_0|$*;*

ii) *For any $A \subset A_0$ with $|A| \geq |A_0|^{13/14}$ and for any finite group G such that there exists a Freiman 7-isomorphism from A onto G, we have $|G| \geq |A|^{15/14}$.*

Let $\alpha \in (0, 1)$ and A_0 be the subset of H

$$A_0 := \{[x, y, z] \mid (x, y, z) \in [0, p^\alpha) \times \mathbb{F} \times \mathbb{F}\}. \tag{4}$$

For p large enough, we plainly have

$$|A_0 \cdot A_0| = 2|A_0| - p^2,$$

thus A_0 is a small squaring subset of H.

Let θ be such that $0 < \theta \leq 1$, on which an additional assumption will be given later. Let A be any subset of A_0 whose cardinality satisfies

$$|A| \geq |A_0|^\theta. \tag{5}$$

By an averaging argument, there exists $x_0, y_0, z_0, z_0', u, v \in \mathbb{F}$ and $X, Y, Z \subset \mathbb{F}$ such that

$$[X, y_0, z_0] \cup [x_0, Y, z_0'] \cup [u, v, Z] \subset A \tag{6}$$

$$|X| \geq \frac{|A|}{p^2}, \quad |Y| \geq \frac{|A|}{p^{1+\alpha}}, \quad |Z| \geq \frac{|A|}{p^{1+\alpha}}. \tag{7}$$

Observe that $|X||Y||Z|^2 \geq p^3$ if

$$|A| \geq p^{(8+3\alpha)/4}, \tag{8}$$

which holds true if we fix α such that

$$\theta = \frac{8+3\alpha}{8+4\alpha}, \tag{9}$$

that is

$$\alpha = \frac{8(1-\theta)}{4\theta-3}, \tag{10}$$

assuming that the following condition on θ holds:

$$\theta \geq \frac{11}{12}.$$

Let $a = [x, y_0, z_0]$, $b = [x_0, y, z_0']$. These are elements of A. Moreover the commutator of a and b is

$$aba^{-1}b^{-1} = [0, 0, xy - x_0y_0].$$

Let $c = [u, v, z]$ and $d = [u, v, z']$ in $[u, v, Z] \subset A$. We thus have

$$aba^{-1}b^{-1}cd^{-1} = [0, 0, xy + z - z' - x_0y_0].$$

For any element t in $\mathbb{F}$, let $N(t)$ be the number of representations of t under the form

$$t = xy + z - z' - x_0y_0, \quad x \in X, \quad y \in Y, \quad z, z' \in Z.$$

One has

$$N(t) = \frac{1}{p} \sum_{h=0}^{p-1} \sum_{\substack{x \in X \\ y \in Y \\ z, z' \in Z}} e\left(\frac{h(xy - x_0y_0 + z - z' - t)}{p}\right),$$

where $e(\alpha)$ is the usual notation for $\exp(2i\pi\alpha)$. We get

$$N(t) \geq \frac{|X||Y||Z|^2}{p} - \frac{1}{p} \sum_{h=1}^{p-1} |S(h)||T(h)|^2,$$

where

$$S(h) = \sum_{(x,y) \in X \times Y} e\left(\frac{hxy}{p}\right), \quad T(h) = \sum_{z \in Z} e\left(\frac{hz}{p}\right).$$

By Vinogradov's inequality

$$|S(h)| \leq \sqrt{p|X||Y|} \quad (\text{if } p \nmid h)$$

and Parseval's identity

$$\frac{1}{p} \sum_{h=1}^{p} |T(h)|^2 = |Z|,$$

we deduce the lower bound

$$N(t) > \frac{|X||Y||Z|^2}{p} - \sqrt{p|X||Y|}|Z|.$$

Hence by (10), $N(t)$ is positive. We thus deduce

$$[0,0,\mathbb{F}] \subset B := A^2A^{-2}AA^{-1}.$$

Let G be any finite group and π any Freiman s-isomorphism from A into G. Our goal is to show that $|G|$ is big compared to $|A|$. We thus may assume that $G = \langle \pi(A) \rangle$.

We assume in the sequel that $s \geq 7$. We start from the property that is proven just above:

$$\pi([0,0,\mathbb{F}]) \subset \pi(B).$$

For any $z \in \mathbb{F}$, we let

$$g_z = \pi([0,0,z]).$$

If $h = \pi([u,v,w]) \in \pi(A)$, then for $s \geq 7$ we have

$$\pi([-u,-v,uv-w+z]) = \pi([u,v,w]^{-1}[0,0,z]) = h^{-1}g_z = g_z h^{-1}. \tag{11}$$

We now show that for some $i \neq j$,

$$g_{\lambda(i-j)} = g_{(\lambda-1)(i-j)}g_{i-j}, \quad 0 < \lambda \leq p.$$

Since $[u,v,Z] \subset A$ and $|Z| > 1$ by (7) and (8), A contains at least two distinct elements $[u,v,i]$ and $[u,v,j]$. We denote $h_k = \pi([u,v,k])$ for $k = i,j$. Since π is a Freiman s-isomorphism from A into G and $s \geq 7$, we get $h_j^{-1}h_i = g_{i-j}$ and by a similar calculation as in (11)

$$g_{(\lambda+1)(i-j)}h_i^{-1} = g_{\lambda(i-j)}h_j^{-1},$$

hence

$$g_{(\lambda+1)(i-j)} = g_{\lambda(i-j)+j}h_j^{-1}h_i = g_{\lambda(i-j)}g_{i-j}.$$

We deduce by induction

$$g_{\lambda(i-j)} = g_{i-j}^{\lambda}, \quad \text{for any } \lambda \geq 1.$$

Thus the order of g_{i-j} in G is either 0 or p. Since $s \geq 2$, we have $h_i \neq h_j$ hence $g_{i-j} = h_j^{-1}h_i \neq 1_G$. This shows that g_{i-j} is of order p in G. We then deduce that p divides the order of G.

Let G_p be the Sylow p-subgroup of G. Since $s \geq 5$ and H is a two-step nilpotent group, G is also a two-step nilpotent group by Property 4 of Section 2. Then by Property 2 of Section 2, G can be written as the direct product $G = G_p \times K$. The projection σ of G onto G_p is a homomorphism thus $\tilde{\pi} = \sigma \circ \pi$ is a Freiman s-homomorphism. Since for $z \neq 0$, h_z has order p in G, $\sigma(h_z)$ has also order p in G_p.

Let $a_1 = [x_1, y_1, z_1]$ and $a_2 = [x_2, y_2, z_2]$ be any elements in A. We have $a_1 a_2 a_1^{-1} a_2^{-1} = [0, 0, x_1 y_2 - x_2 y_1]$. If G_p were abelian we would obtain by using $s \geq 4$

$$1_G = \tilde{\pi}(a_1)\tilde{\pi}(a_2)\tilde{\pi}(a_1)^{-1}\tilde{\pi}(a_2)^{-1} = \tilde{\pi}(a_1 a_2 a_1^{-1} a_2^{-1}) = \tilde{\pi}([0, 0, x_1 y_2 - x_2 y_1]) = \sigma(g_{x_1 y_2 - x_2 y_1}),$$

hence $x_1 y_2 - x_2 y_1 = 0$. We would conclude that $|A| \leq p^2$, a contradiction by the fact that $|A| \geq |A_0|^\theta \geq p^{(2+\alpha)\theta} > p^2$ by (9).

Consequently by Property 3 given in Section 2, G_p is not abelian and $|G_p| \geq p^3$. Finally

$$|G| \geq p^3 = |A_0|^{3/(2+\alpha)} \geq |A|^{(12\theta-9)/2}.$$

The proof of Theorem 3 finishes by choosing the prime p large enough in order to have $|A_0| > n$.

4. **Proof of the main result Theorem 1**

Again, A_0 denotes the set

$$A_0 = \{[x, y, z] \,:\, 0 \leq x < p^\alpha,\ y, z \in \mathbb{F}\},$$

and A any subset of A_0 such that $|A| \geq |A_0|^\theta$. The parameters $\alpha \in (0, 1)$ and $\theta \in (0, 1)$ will be specified below. Again, we have $|A_0| \geq p^{2+\alpha}$ thus

$$|A| \geq p^{(2+\alpha)\theta}. \tag{12}$$

We recall that there exist $x_0, y_0, z_0, z_0', u, v \in \mathbb{F}$ and $X, Y, Z \subset \mathbb{F}$ such that :

$$\begin{gathered} [X, y_0, z_0] \cup [x_0, Y, z_0'] \cup [u, v, Z] \subset A \\ |X| \geq \frac{|A|}{p^2}, \quad |Y| \geq \frac{|A|}{p^{1+\alpha}}, \quad |Z| \geq \frac{|A|}{p^{1+\alpha}}. \end{gathered} \tag{13}$$

For $(x, y, z) \in X \times Y \times Z$, one has

$$[x, y_0, z_0][x_0, y, z_0'][x, y_0, z_0]^{-1}[x_0, y, z_0']^{-1}[u, v, z] = [u, v, xy + z - x_0 y_0].$$

Our first goal is to show that $[u, v, t]$ is in $A^2 A^{-2} A$ except for t belonging to a small subset E of exceptions.

First step: For any t in $\mathbb{F}$, let $r(t)$ be the number of triples $(x, y, z) \in X \times Y \times Z$ such that

$$t = xy + z - x_0 y_0.$$

One cannot prove that $r(t) > 0$ for any t. Nevertheless, we will show that except for a small part of elements t, this property holds. Let C be the set of those elements of t for which

$r(t) > 0$. Then by the Cauchy-Schwarz inequality

$$|C| \geq \frac{(|X||Y||Z|)^2}{\sum_t r(t)^2}. \tag{14}$$

Furthermore $\sum_t r(t)^2$ coincides with the number of solutions of

$$xy + z = x'y' + z', \quad x, x' \in X,\ y, y' \in Y,\ z, z' \in Z.$$

If we fix $x = x_1$, $x' = x'_1$ and $z' = z'_1$, it gives the equation of an hyperplan D_{x_1,x'_1,z'_1} in $\mathbb{F}^3$:

$$x_1 y - x'_1 y' + z - z'_1 = 0.$$

All these hyperplanes are different and there are $|X|^2|Z|$ such hyperplanes. The possible number of points $(y, y', z) \in Y \times Y \times Z$ is $|Y|^2|Z|$.

In [7], L.A. Vinh established a Szemeredi-Trotter type result by obtaining an incidence inequality for points and hyperplanes in $\mathbb{F}^d$. It is connected to the Expander Mixing Lemma (see Corollary 9.2.5 in [1]). We have:

Lemma 5 (L.A. Vinh [7])**.** *Let $d \geq 2$. Let $\mathcal{P}$ be a set of points in $\mathbb{F}^d$ and $\mathcal{H}$ be a set of hyperplanes in $\mathbb{F}^d$. Then*

$$|\{(P, D) \in \mathcal{P} \times \mathcal{H} \ :\ P \in D\}| \leq \frac{|\mathcal{P}||\mathcal{H}|}{p} + (1 + o(1))p^{(d-1)/2}(|\mathcal{P}||\mathcal{H}|)^{1/2}.$$

By this result with $d = 3$, we get for any large p

$$\sum_t r(t)^2 \leq \frac{(|X||Y||Z|)^2}{p} + 2p|X||Y||Z|,$$

which yields by (14)

$$|C| \geq p - \frac{2p^3}{|X||Y||Z|}.$$

Thus the set E of exceptions $t \in \mathbb{F}$ with $r(t) = 0$ has cardinality

$$|E| \leq \frac{2p^3}{|X||Y||Z|}. \tag{15}$$

Second step: We fix z_1 any element in Z and let $Z_1 = Z \setminus \{z_1\}$. For any $z \in Z_1$, we denote

$$m(z) = \max\{m \leq p \ :\ z_1 + j(z - z_1) \notin E,\ 2 \leq j \leq m\}$$

if the maximum exists and we let $m(z) = p$ otherwise. Let

$$T = \left[\frac{|Z_1|}{2|E|}\right] \tag{16}$$

If we denote by Z'_1 the set of the elements $z \in Z_1$ with $m(z) \leq T$, then

$$|Z'_1| = \sum_{m \leq T} |\{z \in Z_1 \ :\ m(z) = m\}| \leq \sum_{m \leq T} |E| \leq \frac{|Z_1|}{2},$$

since $m = m(z)$ implies $z_1 + (m+1)(z - z_1) \in E$. It follows that $m(z) > T$ for at least one half of the elements z in Z_1. We denote by $\tilde{Z}_1$ the set of those elements z. We have

$$|\tilde{Z}_1| \geq \frac{|A|}{2p^{1+\alpha}}. \tag{17}$$

Lemma 6. *Assume that* $23/24 < \theta \leq 1$ *and let* γ *be a positive real number such that*

$$\gamma < \frac{2(2+\alpha)\theta - (3+2\alpha)}{3}. \tag{18}$$

If $|E| < p^{\gamma}$*, then there exists an integer* t *with* $1 \leq t \leq T$ *and two distinct elements* $z, z' \in \tilde{Z}_1$ *such that*

$$z' - z \notin E - E \quad \textit{and} \quad z' = z_1 + t(z - z_1) \tag{19}$$

Proof. For $1 \leq t \leq T$, we denote by $s(t)$ the number of pairs z, z' of elements of $\tilde{Z}_1$ with the required property. It is sufficient to show that

$$\sum_{t=1}^{T} s(t) > 0.$$

This sum can be rewritten as

$$\sum_{t=1}^{T} \frac{1}{p} \sum_{0 \leq |h| \leq p/2} \sum_{\substack{z,z' \in -z_1 + \tilde{Z}_1 \\ z'-z \notin E-E}} e\left(\frac{h(z^{-1}z' - t)}{p}\right).$$

The contribution related to $h = 0$ is plainly bigger than

$$\frac{T}{p}(|\tilde{Z}_1|^2 - |\tilde{Z}_1||E - E|),$$

thus

$$\sum_{t=1}^{T} s(t) \geq \frac{T}{p}(|\tilde{Z}_1|^2 - |\tilde{Z}_1||E - E|) - \frac{1}{p} \sum_{0 < |h| < p/2} \Big| \sum_{t=1}^{T} e\left(\frac{-th}{p}\right) \Big| \Big| \sum_{\substack{z,z' \in -z_1 + \tilde{Z}_1 \\ z'-z \notin E-E}} e\left(\frac{hz^{-1}z'}{p}\right) \Big|.$$

By extending the summation over z and z', we obtain for any $h \neq 0$

$$\Big| \sum_{\substack{z,z' \in -z_1 + \tilde{Z}_1 \\ z'-z \notin E-E}} e\left(\frac{hz^{-1}z'}{p}\right) \Big| \leq \Big| \sum_{z,z' \in -z_1 + \tilde{Z}_1} e\left(\frac{hz^{-1}z'}{p}\right) \Big| + |\tilde{Z}_1||E - E|,$$

which is less than or equals to

$$(\sqrt{p} + |E - E|)|\tilde{Z}_1|$$

by using Vinogradov's inequality for the estimation of the sum over z and z'. Hence by the bounds

$$\Big| \sum_{t=1}^{T} e\left(\frac{-ht}{p}\right) \Big| \leq \frac{p}{2|h|} \quad \text{for } 0 < |h| < p/2,$$

and

$$\sum_{h=1}^{(p-1)/2} \frac{1}{h} \le \ln p,$$

we get

$$\sum_{t=1}^{T} s(t) \ge \frac{T}{p}(|\tilde{Z}_1|^2 - |\tilde{Z}_1||E-E|) - (\sqrt{p} + |E-E|)|\tilde{Z}_1| \ln p.$$

From the trivial bound $|E-E| \le |E|^2$ and by (16) and (17), this sum is positive whenever $|E| \le p^{\gamma}$ for p is large enough, where γ is any positive number such that

$$\gamma < \min\left(\frac{(2+\alpha)\theta - (1+\alpha)}{2}; \frac{4(2+\alpha)\theta - (7+4\alpha)}{2}; \frac{2(2+\alpha)\theta - (3+2\alpha)}{3}\right). \tag{20}$$

The second argument in this minimum is less than or equal to the first since $\theta \le 1$ and the third is less than the second since $\theta > 23/24$. Thus condition (20) reduces to (18), and the lemma follows. □

By (13) and (15), we deduce from the lemma that the condition

$$7 + 2\alpha - 3(2+\alpha)\theta < \frac{2(2+\alpha)\theta - (3+2\alpha)}{3},$$

is sufficient in order to ensure that system (19) has at least one solution, assuming p is large enough. This condition reduces to

$$\theta > \frac{24+8\alpha}{22+11\alpha}$$

or equivalently

$$\alpha > \alpha_0(\theta) := \frac{24-22\theta}{11\theta-8}. \tag{21}$$

Since $\alpha < 1$, we must choose θ such that $\theta > \frac{32}{33}$. Fixing

$$\alpha = \alpha_0(\theta) + \varepsilon, \tag{22}$$

this yields

$$p^3 \ge |A|^{3/(2+\alpha)} \ge |A|^{3(11\theta-8)/8-\varepsilon}, \tag{23}$$

for any $p \ge p_0(\epsilon)$. For $\theta = 43/44$, it will give the desired exponents in Theorem 1.

Third step: We have at our disposal $z_1, z \in Z$ and $t \in \mathbb{F}$ such that

$$z_1 + j(z - z_1) \notin E, \quad j = 2, \dots, t, \quad \text{and} \quad z_1 + t(z - z_1) \in Z. \tag{24}$$

Let $\pi : A \to G$, where G is a finite group, be a Freiman 6-isomorphism. As in the proof of Theorem 3, we will show that p divides $|G|$ and that the p-Sylow subgroup of G cannot be abelian. It will ensure the bound $|G| \ge p^3$ and the theorem will follow by (23).

Let

$$h = \pi([0,0,z-z_1]) = \pi([u,v,z_1])^{-1}\pi([u,v,z]). \tag{25}$$

Let us show that for any j such that $j(z-z_1)+z_1 \notin E$, we have $\pi([0,0,j(z-z_1)]) = h^j$.

If $1 \le j \le t$, we proceed by induction: for $j=1$, the property is plainly true. Let $2 \le j \le t$. We have

$$\pi([u,v,j(z-z_1)+z_1][u,v,z]^{-1}) = \pi([u,v,(j-1)(z-z_1)+z_1][u,v,z_1]^{-1}).$$

By (24) and by definition of E, both elements $[u,v,(j-1)(z-z_1)+z_1]$ and $[u,v,j(z-z_1)+z_1]$ belong to $A^2A^{-2}A$. Moreover $[u,v,z],[u,v,z_1] \in A$ hence, by the fact that π is a Freiman 6-homomorphism, we get

$$\pi([u,v,j(z-z_1)+z_1])\pi([u,v,z])^{-1} = \pi([u,v,(j-1)(z-z_1)+z_1])\pi([u,v,z_1])^{-1}.$$

Thus, by (25)

$$\pi([u,v,j(z-z_1)+z_1]) = \pi([u,v,(j-1)(z-z_1)+z_1])h.$$

By multiplying on the left by $\pi([u,v,z_1])^{-1}$ and using again that π is a Freiman 6-homomorphism, we get

$$\pi([0,0,j(z-z_1)]) = \pi([0,0,(j-1)(z-z_1)])h = h^j$$

by the induction hypothesis.

For larger j, we again induct: let $j>t$ be such that $j(z-z_1)+z_1 \notin E$. Then at least one of the two elements $(j-1)(z-z_1)+z_1$ or $(j-t)(z-z_1)+z_1$ is not in E since $z'-z \notin E-E$.

If $(j-1)(z-z_1)+z_1 \notin E$ we argue by induction as above. If $(j-t)(z-z_1)+z_1 \notin E$ we slightly modify the argument: since

$$\pi([u,v,j(z-z_1)+z_1][u,v,t(z-z_1)+z_1]^{-1}) = \pi([u,v,(j-t)(z-z_1)+z_1][u,v,z_1]^{-1})$$

and π a Freiman 6-isomorphism, we get

$$\begin{aligned}\pi([u,v,j(z-z_1)+z_1]) &= \pi([u,v,(j-t)(z-z_1)+z_1])\pi([u,v,z_1])^{-1}\pi([u,v,t(z-z_1)+z_1]) \\ &= \pi([u,v,(j-t)(z-z_1)+z_1])h^t,\end{aligned}$$

and finally by induction

$$\pi([0,0,j(z-z_1)]) = \pi([u,v,(j-t)(z-z_1)+z_1])h^t = h^{j-t}h^t = h^j.$$

Since $z_1 \notin E$, we obtain $h^p = 1$ in G, thus either $h=1$ or h has order p. But $z \neq z_1$ hence $[0,0,z-z_1] = [u,v,z][u,v,z_1]^{-1} \neq 1_H$, hence $h \neq 1_G$ since π is a Freiman 6-isomorphism. We deduce that G admits an element of order p, thus the p-Sylow subgroup G_p of G is not trivial. By considering the canonical homomorphism $\sigma : G \to G_p$, $\tilde{\pi} = \sigma \circ \pi$ is a Freiman 6-homomorphim of A onto G_p. Hence for any $a = [x, y, z]$ and $b = [x', y', z']$ in A

$$[\tilde{\pi}(a); \tilde{\pi}(b)] = \tilde{\pi}([a; b]) = \tilde{\pi}([0, 0, xy' - x'y])$$

which must be equal to 1_G if G_p is assumed to be abelian. It would mean that (x, y) belongs to a single line for any $[x, y, z] \in A$, giving $|A| \le p^2$ a contradiction to

$$\frac{\ln |A|}{\ln p} \ge \theta(2 + \alpha) > \theta(2 + \alpha_0(\theta)) = \frac{8\theta}{11\theta - 8} > 2,$$

obtained by (12), (21) and (22).

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