Additive Structure of Difference Sets

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Abstract: Additive structure of difference and iterated difference sets are investigated. In the survey we collect results and some applications of theorems of Bogolyubov and Følner. Some exercises are also included

1. Introduction

Let A be a subset of integers with positive upper density,

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{A(n)}{n} > 0,$$

where $A(n) := \sum_{\substack{a\in A\\ 1\le a\le n}} 1$. The difference set is defined as follows:

$$D(A) = \{a - a' : a, a' \in A\}.$$

The iteration of this operation (i.e. the second difference set) is

$$D_2 = D(D(A)) = A - A + A - A,$$

and generally for $k > 1$,

$$D_{k+1} = D(D_k(A)).$$

Define the time of stability of A by $T(A) = \min\{k \mid D^+_{k+1}(A) = D^+_k(A)\}$, where the operation $D^+(\cdot)$ takes just the positive part of $D(\cdot)$.

Stewart and Tijdeman and later Ruzsa proved

$$T(A) \le 2 + \log_2(\overline{d}(A)^{-1} - 1).$$

Let G be a countable torsion group and let $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \subseteq \cdots$ be a sequence of finite subgroups of G. Then G is said to be σ-finite with respect to $\{H_n\}$ if $G = \bigcup_{n=1}^{\infty} H_n$.

Let $A \subseteq G$. The asymptotic upper density of A is defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap H_n|}{|H_n|}. \tag{1}$$

We can introduce the time of stability in groups as well.

Assume the sequence $\{D_k(A)\,;\, k \geq 0\}$ is stable (i.e. for some k, $D_{k+1}(A) = D_k(A)$). Let $T(A,G)$ be the time of stability defined by

$$T(A,G) = \min\{k \mid D_{k+1}(A) = D_k(A)\}.$$

Theorem ([H-2001]) *Let G be a σ-finite abelian group with respect to $\{H_n\}$ and let A be a non empty subset of G. Let $\overline{d}(A)$ be the upper density of A defined by* (1). *If $\overline{d}(A) > 0$, then*

$$T(A,G) \leq \log_2(\overline{d}(A)^{-1}) + 2.$$

F. Hennecart and the author extended this theorem to finite groups and σ-finite groups (not necessary abelian). For an arbitrary finite groups:

Theorem ([H-H 2007])

Let A be a generating subset of a finite group G such that $1 \in A$. Let k_0 defined by

$$k_0 = \begin{cases} 1 & \text{if } |G|/2 < |A| \leq |G|, \\ 2 & \text{if } |G|/3 < |A| \leq |G|/2, \\ \left\lfloor \log_2\left(\frac{2|G|}{3|A|} - 1\right)\right\rfloor + 3 & \text{if } |A| \leq |G|/3. \end{cases}$$

Then, for any integer $k \geq k_0$

$$D_k(A) = G. \tag{2}$$

For σ-finite groups:

Theorem ([H-H 2007])

Let G be a σ-finite group with respect to $\{H_n\}$ and let A be a non empty subset of G. Assume that A has a positive upper density and $\alpha := \overline{d}(A)^{-1} \geq 3$. Then

$$T(A,G) \leq \lfloor \log_2(2\alpha/3 - 1)\rfloor + 3, \tag{3}$$

where $\lfloor u \rfloor$ denotes the greatest integer less than or equal to the real number u.

Exercises

1. For $A \subseteq \mathbb{Z}$, $\overline{d}(A) := \limsup_{n\to\infty} \frac{A(n)}{n} > 0$, we have that $D(A)$ has bounded gaps, i.e. writing $D(A) = \{d_1 < d_2 < \cdots < d_k < \dots\}$ there exists a $K > 0$, for which $d_{k+1} - d_k \le K$.

The proof will be given in a more general structure (see the proof of the Lemma in section 2)

2. Prove that for $A \subseteq \mathbb{Z}$, $\overline{d}(A) := \limsup_{n\to\infty} \frac{A(n)}{n} > 0$, that for every $k \in \mathbb{N}$ the sequence $D(A)$ contains a $k-$terms arithmetic progression.

3. Nevertheless the size of the gaps is uncertain;
for every $\delta > 0$ and K there exists a set $A \subseteq \mathbb{Z}$, $\overline{d}(A) := \limsup_{n\to\infty} \frac{A(n)}{n} = \delta$, and $\max_k d_{k+1} - d_k \ge K$.

4. Let G be an Abelian group and assume that the sequence $\{D_k(A)\}$ is stable, i.e. there exists a k_0, such that $k \ge k_0$

$$D_{k+1} = D_k(A).$$

Show that $D_k(A)$ is a subgroup in G.

5. Let $m \in \mathbb{N}$, $A = \{x : x \equiv 0, 1(\mod m)\} \subseteq \mathbb{N}$.
Prove that $d(A)$ exists where $d(A)$ is the density of a sequence defined by $\lim_{n\to\infty} \frac{A(n)}{n}$ and find its value. Determine $T(A)$, the time of the stability as well.

2. The case D(A)

V. Bergelson investigated the additive structure of $D(A)$.

Let us define first the Banach density of set of integers as follows:

$$d^*(A) := \sup\{L : \forall m,\ \exists (a_m, b_m)\ |b_m - a_m| \ge m, and\ \frac{|A \cap (a_m, b_m)|}{|b_m - a_m|} \ge L\}.$$

(We shall define $d^*(A)$ in higher dimension as well)

Exercises

Prove that $\overline{d}(A) \leq d^*(A)$.

Bergelson proved

Theorem ([B 1985])
There exists an infinite set B of integers for which

$$A - A \supseteq B + B + \cdots + B = B \cdot k,$$

provided A has positive Banach density.

The proof of this theorem based on an ergodic theorem (Fürstenberg Correspondence Principle).

Theorem ([F 1977])

Given a set $E \subseteq \mathbb{Z}$, with positive upper density, there exists a probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$, $\mu(A) = \overline{d}(E)$, such that any $k \in \mathbb{N}$, $n_1, n_2, \dots, n_k \in \mathbb{Z}$, one has

$$d^*(E \cap (E - n_1) \cap (E - n_2) \cap \cdots \cap (E - n_k)) = \mu(A \cap T^{-n_1} A \cap T^{-n_2} A \cap \cdots \cap T^{-n_k} A).$$

$(X, \mathcal{B}, \mu, T)$ is a probability measure preserving system, where

X is a set

$\mathcal{B}$ is a $\sigma-$algebra over X

$\mu(\cdot)$ is a probability measure

$T : X \mapsto X$, measurable transformation; i.e. for all $A \in \mathbf{B}$, $\mu(T^{-1}A) = \mu(A)$.

Now we give a pure combinatorial proof for Bergelson's theorem in higher dimension as well.

Definition:

Let $A \subseteq \mathbb{Z}^n$, the counting function is defined by

$$A(x) = \sum_{a \in A; |a| \leq x} 1,$$

where $|a|$ is the length of a (the distance from the origin).
Define the *discrete* rectangle of $\mathbb{Z}^n$ by

$$R = [a_1, b_1] \times [a_2, b_2] \times \dots [a_n, b_n] \cap \mathbb{Z}^n.$$

The volume of R is $|R| = \prod_i (b_i - a_i + 1)$.
An upper Banach density of A now is

$$d^*(A) := \sup\{L : \forall m,\ \exists R_m,\ \min_i |b_i - a_i| \geq m, s.t. \frac{|A \cap R_m|}{|R_m|} \geq L\}.$$

We formulate the theorem now;

Theorem ([H-2002])

Let $A \subseteq \mathbb{Z}^n$, with $d^*(A) = \gamma > 0$. For every integer M there is an infinite set $B \subseteq \mathbb{Z}^n$ such that

$$D(A) \supseteq = B \cdot M := B + B + \cdots + B(M\ times).$$

Proof:

$A - A$ is symmetric to the origin w.l.g. assume $M > 0$. Consider the lattice points of cube $\{\mathbf{x}_i\}_{i=1}^{M^n}$; $\mathbf{x}_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_n}); 0 \leq x_{i_j} \leq M - 1$. Write

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \equiv \mathbf{v} = (v_1, v_2, \ldots, v_n)(modM)$$
$$\Leftrightarrow$$

$$u_i \equiv v_i(modM);\ \forall i;\ 1 \leq i \leq n.$$

Let

$$A_i = \{\mathbf{a} \in A : \mathbf{a} \equiv \mathbf{x}_i(modM)\}.$$

$$d^*(A) = \gamma > 0\ \Rightarrow\ d^*(A_i) = \rho > 0$$

for some i.

Let

$$A' = A_i - \mathbf{x}_i \subseteq L := \{\mathbf{u} \equiv \mathbf{0}(modM)\}.$$

Surely

$$A' - A' = A_i - A_i \subseteq A - A.$$

Lemma:

There exists a finite set U such that $A' - A' + U = L$.

Proof of the Lemma:

Let $U = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r, \dots\}$ be the maximal subset of $\mathbb{Z}^n$, such that the sets

$$\mathbf{u}_1 + A', \mathbf{u}_2 + A', \dots, \mathbf{u}_r + A', \dots$$

are pairwise disjoint.

$r \leq 4/\rho$.

Since $d^*(A') = d^*(A_i) = \rho > 0$, there is a rectangle R such that $|R \cap A'| \geq \frac{\rho|R|}{2}$. Assume that the minimal length of edge of R is large enough, we get

$$|R| \geq |R \cap \{(\mathbf{u}_1 + A') \cup \dots \cup (\mathbf{u}_r + A')\}| =$$

$$|R \cap (\mathbf{u}_1 + A')| + \dots + |R \cap (\mathbf{u}_r + A')| \geq r\frac{|R \cap A'|}{2} \geq r\frac{\rho|R|}{4}$$

which gives $r \leq 4/\rho$.

If U would not be fulfil the statement of the lemma, then there will be an $x \in L \setminus (A' - A' + U)$, for which

$$x \notin A' - A' + \mathbf{u}_i$$

for $i = 1, 2, \dots r$, or or equivalently

$$x + A' \cap \mathbf{u}_i + A' = \emptyset.$$

It contradicts to the fact that U is the maximal respect to it.

We introduce an $r-$coloring

$$\chi(\mathbf{x}_1, \dots, \mathbf{x}_M) \mapsto \{1, 2, \dots, r\}$$

of all M element subsets of L as follows: for an $M-$tuple $\mathbf{x}_1, \dots, \mathbf{x}_M$ the color of it determined by

$$\chi(\mathbf{x}_1, \dots, \mathbf{x}_M) \in \{i : \mathbf{x}_1 + \dots + \mathbf{x}_M \in A' - A' + \mathbf{u}_i\}.$$

(Note the coloring is not necessary unique, if it is not then use an arbitrary color)

Lemma:(Ramsey)

Let X be a countable set and color of all $M-$tuples of X by r colors (color the $M-$uniform graph of X).
There exists an $infinite$ set B' which is monochromatic.

Now by the lemma we have there is an infinite set B' which is monochromatic i.e. for every $M-$tuple $\mathbf{x}_1, \dots, \mathbf{x}_M$ in B'

$$\mathbf{x}_1 + \dots + \mathbf{x}_M \in A' - A' + \mathbf{u}_s,$$

for some fixed s.

Finally let

$$B := B' - \frac{\mathbf{u}_s}{M}.$$

$$\mathbf{u}_s \in L \;\Rightarrow\; \frac{\mathbf{u}_s}{M} \in \mathbb{Z}^n.$$

$$A' - A' + \mathbf{u}_s \supseteq B' + B' + \cdots + B'(M\ times) = (B + \frac{\mathbf{u}_s}{M}) + \cdots + +(B + \frac{\mathbf{u}_s}{M})(M\ times) =$$

$$= B + B + \cdots + B + M\frac{\mathbf{u}_s}{M},$$

which implies

$$A - A \supseteq A' - A' \supseteq B + B + \cdots + B(M\ times).$$

3. An intermezzo; D(D(A)) is highly well-structured

On a Bohr set we mean the set

$$B(S, \varepsilon) = \{m \in \mathbb{Z} : \max_{s \in S} \|sm\| < \varepsilon\},$$

where S is a finite subset of the set of real numbers and $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$.

One of a central observation in Additive Combinatorics

Theorem ([Bo-1939])

Let A be a subset of integers, $\overline{d}(A) := \gamma > 0$.
There exists a Bohr set $B(S, \varepsilon)$ containing in $D_2(A)$, in fact S, and ε can be chosen to

$$|S| \le \frac{1}{\gamma^2}; \quad \varepsilon = \frac{1}{4}.$$

One can ask:

What about the first difference sets?

4. First difference sets and Bohr sets I

In this section we investigate two questions:

1. Is there a Bohr set containing in $A - A$ provided A has positive upper density?

2. One can cover $A - A$ by a "small" Bohr set?

For the first question the answer is negative:

Theorem

There exists a set A of positive integers with positive density for which the first difference set $A - A$ does not contain any Bohr set.

Proof:

For the proof we need some notion from the graph theory:

A graph G is defined on the set $\mathbb{Z}$ of integers; i.e. $G = G(V, E)$, where the set of vertices V is the set of integers, $E \subseteq \mathbb{Z} \times \mathbb{Z}$.

• G is said to be *shift invariant*

$$(x, y) \in E \quad \Leftrightarrow \quad (x + n, y + n) \in E,$$

for every $n \in \mathbb{Z}$.

• The chromatic number $\chi(G) = r$ of a graph G is the smallest number r for which there exists a coloring of V by r colors s.t. for every $x, y \in V$, $\chi(x) \neq \chi(y)$, if $(x, y) \in E$. If there is no such a number then $\chi(G) = \infty$.

• A set $S \subseteq V$ is said to be *independent* if

$$(S \times S) \cap E = \emptyset,$$

i.e. there is no edge in the set S.

We need a result of *Křiž*:

Lemma:

For every $\varepsilon > 0$ there exists a shift invariant graph G on $\mathbb{Z}$ with chromatic number $\chi(G) = \infty$ and containing an independent set A of dense $> 1/2 - \varepsilon$.

From this lemma we derive that for this given independent set A:

$A - A$ does not contain any Bohr set.

Indeed since A is independent set and the graph is shift invariant we conclude for every u, w

$$u - w \in A - A \quad \Rightarrow \quad (u, w) \notin E(G). \tag{$*$}$$

(there exists an integer $x \in \mathbb{Z}$ s.t. $(u - x, w - x) \in A$.)

Contrary assume there exists a Bohr set

$$B(S, \varepsilon) \subseteq A - A.$$

Let $|S| = m$, and split the $m-$dimensional cube $[0, 1]^m$ into subcubes $C_1, C_2, \ldots, C_s$ s.t. the sides of $C_i < \varepsilon/2$.

Color $\mathbb{Z}$ by the following way:

$$Z_i := \{n : \|sn\| \in C_i; \forall s \in S\}.$$

It implies that for every i

$$\cup_{i=1}^{s} Z_i = \mathbb{Z}; \quad \text{and} \quad Z_i - Z_i \subseteq B(S, \varepsilon),$$

i.e. we have an s coloring of $\mathbb{Z}$ for which $Z_i - Z_i \subseteq B(S, \varepsilon) \subseteq A - A$. By $(*)$ we conclude that for every $u, w \in Z_i$ $(u, v) \notin E(G)$. Hence

$$s \geq \chi(G) = \infty$$

a contradiction.

For the second question we investigate just the modular version. (The case of integers comes in the similar way as in the proof of Bogolyubov's theorem). The following theorem is a version of a theorem of Tao and Vu.

Theorem([TV, 2007])

Let $\gamma > 0$, $A \subseteq \mathbb{Z}_N$, $|A| > \gamma N$. There exits a Bohr set $B(S, \varepsilon)$, which covers $A - A$, $|S| \leq (1 - \alpha)^{-2}$, and

$$\varepsilon = \sqrt{\frac{\alpha}{2\gamma}}.$$

Proof:

Let as usual $\widehat{B}(r) = \sum_x B(x) e(rx)$, where $B(x)$ the indicator of the set B. Denote $D = A - A$.

Let

$$S = \{r : |\widehat{D}(r)| > (1 - \alpha)N\},$$

$\alpha > 0\}$.

By the Parseval formula we conclude that $|S| \leq (1-\alpha)^{-2}$.

Let $r \in S$. Now there exits an $u \in \mathbb{Z}_N$, such that

$$\Re \sum_{z \in D} e(rz+u) \geq (1-\alpha)N,$$

thus

$$\sum_{z \in D} (1 - \Re e(rz+u)) \leq \alpha N.$$

Fix an $x, y \in A$, and since the terms are non-negative we can write

$$\sum_{a \in A} |1 - \Re e(r(x-a)+u)| \leq \alpha N,$$

and

$$\sum_{a \in A} |1 - \Re e(r(y-a)+u)| \leq \alpha N.$$

Use for the them the Cauchy-Schwarz inequality we get

$$\sum_{a \in A} |1 - \Re e(r(x-a)+u)|^{1/2} \leq \sqrt{|A|} \sqrt{\sum_{a \in A} |1 - \Re e(r(x-a)+u)|} \leq \sqrt{|A|}\sqrt{\alpha}\sqrt{N} = \sqrt{\alpha\gamma}N$$

and a similar bound for y.

Exercise:

Prove

$$|1 - e(\alpha)| \leq \sqrt{2}|1 - \Re e(\alpha)|^{1/2}.$$

By this exercise we obtain

$$\sum_{a \in A} |1 - e(r(x-a)+u)| \leq \sqrt{2\alpha\gamma}N$$

and

$$\sum_{a \in A} |1 - e(r(y-a)+u)| \leq \sqrt{2\alpha\gamma}N.$$

Now by the triangle inequality we have

$$\sum_{a} |e(r(x-a)+u) - e(r(y-a)+u)| \leq 2\sqrt{2\alpha\gamma}N.$$

The quantities $|e(r(x-a)+u) - e(r(y-a)+u)|$ is the same than $|1 - e(r(x-y))|$, and using $4\|z\| \leq |1 - e(z)|$, finally we obtain

$$4\gamma N\|r(x-y)\| \leq |A|\|r(x-y)\| = \sum_a |1-e(r(x-y))| \leq 2\sqrt{2\alpha\gamma}N,$$

rearranging we have

$$\|r(x-y)\| \leq \frac{\sqrt{2}}{2}\sqrt{\frac{\alpha}{\gamma}},$$

i.e. for every $x, y \in A$, $x - y \in B(S, \varepsilon)$, where

$$\varepsilon = \sqrt{\frac{\alpha}{2\gamma}}.$$

5. Raimi's Theorem; Difference Set of Partitions

A branch of combinatorial analysis – called Ramsey theory – investigates partitions of certain structures.

Int his section we investigate a partition version of difference sets. One of it Raimi's theorem (for which the simplest proof is due to Hindman):

Theorem 5.1. *There exists $E \subseteq \mathbb{N}$ such that, whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r D_i$ there exist $i \in \{1, 2, \ldots, r\}$ and $k \in \mathbb{N}$ such that $(D_i + k) \cap E$ is infinite and $(D_i + k) \setminus E$ is infinite.*

One can read this theorem as follows:

Theorem 5.2. *There exists a partition of $\mathbb{N} = E_1 \cup E_2$ such that, whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r D_i$ there exist $i \in \{1, 2, \ldots, r\}$ and $k \in \mathbb{N}$ such that both $k \in E_1 - D_i$ and $k \in E_2 - D_i$ hold infinitely many times.*

A generalization of it is the following

Theorem 5.3. *([H-2005])*

Let $r \in \mathbb{N}$ and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive real numbers such that $\sum_{i=1}^r \alpha_i = 1$. There exists a disjoint partition $\mathbb{N} = \bigcup_{i=1}^r E_i$ such that

(1) for every $i \in \{1, 2, \ldots, r\}$, $d(E_i) = \alpha_i$ and

(2) for each $t \in N$ and each partition $\mathbb{N} = \bigcup_{j=1}^{t} F_j$, there exist $m \in \{1, 2, \ldots, t\}$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that for every $h \in FS(\{x_n\}_{n=1}^{\infty})$ and every $i \in \{1, 2, \ldots, r\}$, $(F_m + h) \cap E_i$ is infinite.

Proof. (Sketch)

We prove the simplest case when $r = 2$ and $\alpha_1 = \alpha_2 = 1/2$. The proof of the general case is similar just more technical.

Now color the unit interval $[0, 1)$ as follows: the first half by two colors: $[0, 1/4)$ is red $[1/4, 1/2)$ is blue. Color the half of the rest again by two colors; i.e. $[1/2, 5/8)$ is red, and $[5/8, 3/4)$ is blue. Again color the half of the rest by two colors e.t.c. So we have an infinite sets of red intervals with total length

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2},$$

and blue intervals with the same length.

Now our partition will be the following: Let γ be a positive irrational. Let

$$E_1 = \{x \in \mathbb{N} : \langle \gamma x \rangle \in \text{some red interval}\}$$

and

$$E_2 = \{x \in \mathbb{N} : \langle \gamma x \rangle \in \text{some blue interval}\}.$$

Let $t \in N$ and let $\mathbb{N} = \bigcup_{j=1}^{t} F_j$.

Exercise :

1. Prove that $d(E_i) = \alpha_i = 1/2$.

2. For any c, d with $0 \leq c < d \leq 1$ there exists $m \in \{1, 2, \ldots, t\}$ and there exist a, b, with $c \leq a < b \leq d$ such that $\{\langle \gamma x \rangle : x \in F_m\}$ is dense in (a, b).

The sequence $\{x_n\}_{n=1}^{\infty}$ will be defined inductively;

Since the length of the red-blue pairs of intervals tends to zero and since the sequence $\{\langle \gamma m \rangle\}$ $(m \in \mathbb{N})$ is uniformly distributed, we get that there exists an integer x_1, such that $\{\langle \gamma(x+x_1) \rangle : x \in F_m\} \cap (a, b)$ covers a red-blue pairs of intervals (there are intervals with length $1/2^{k+1} < b - a$). It means that there are infinitely many coincidence of the sets both of $x_1 + F_m$, E_1, and $x_1 + F_m$, E_2.

Assume that a sequence $\{x_n\}_{n=1}^{N}$ has been defined. Since

$$FS(\{x_n\}_{n=1}^{N+1} = FS(\{x_n\}_{n=1}^{N}) + \{0, x_{N+1}\},$$

and again since $\{\langle \gamma m \rangle\}$ $(m \in \mathbb{N})$ is uniformly distributed, we obtain that there exists an integer x_{N+1} such that $\{\langle \gamma(\sum_{i=1}^{N+1} x_i + x) \rangle : x \in F_m\} \cap (a, b)$ covers the give red-blue pairs of intervals.

□

6. First difference sets and Bohr sets II; Følner's Theorem

Recall on a Bohr set we mean the set

$$B(S,\varepsilon) = \{m \in \mathbb{Z} : \max_{s\in S} \|sm\| < \varepsilon\},$$

where S is a finite subset of the set of real numbers.

Følner proved

Theorem 6.1. *Assume that $A \subseteq \mathbb{N}$ and $\overline{d}(A) > 0$. There exists a Bohr set $B(S,\varepsilon)$ such that*

$$B(S,\varepsilon) \setminus (A - A)$$

has density 0.

Exercises:
1. Prove

$$d(B(S,\varepsilon)) \geq \varepsilon^{|S|}.$$

2. Prove that a Bohr set $B(S,\varepsilon)$ has bounded gaps.

7. Applications of Følner theorem

(Joint work with Imre Ruzsa. The results will be published in another place [H-R 2007].)

As in the introduction we mentioned $D(D(A))$ always contains a Bohr set, while the set $D(A)$ not necessary contains a Bohr set. Now we investigate the three-fold sum-differences of A.

Theorem 7.1. *There is a symmetric set A of integers such that $0 \in A$, the positive elements of A form a set of positive density and the set $A + A + A$ does not contain a Bohr set.*

On the other hand we prove that $A + A - A$ is always a Bohr neighborhood of some $a \in A$.

Theorem 7.2. *Assume that $\overline{d}(A) > 0$. There exists a subset A' of A, $\overline{d}(A') > 0$, such that for every $a' \in A'$, the set $A + A - A - a'$ contains a Bohr set.*

Corollary 1. *Følner theorem implies Bogolyubov theorem*

Proof. Since for $a' \in A' \subseteq A$, the set $A + A - A - a'$ contains a Bohr set and by $A + A - A - a' \subseteq A - A + A - A$, we get Bogolyubov's result. □

By the Følner's theorem can be proved a proof of Bergelson's theorem.

Theorem 7.3. *Assume* $\overline{d}(A) > 0$. *There exists an infinite set* C *such that*

$$A - A \supseteq FS(C) \cup FP(C),$$

where $FS(C) = \{\sum_{x \in X} : X \subseteq C;\ X \textit{is finite}\}$, $FP(C) = \{\prod_{x \in X} : X \subseteq C;\ X \textit{is finite}\}$.

References

[H-2001] N. Hegyvári, On iterated difference sets in groups, Period. Math. Hungar. 43 (2001), no. 1-2, 105–110.

[H-H 2007] N. Hegyvári & F. Hennecart, Iterated Difference Sets in Sigma-finite Groups (to appear)

[B-1985] V. Bergelson, Sets of reccurence of $\mathbb{Z}^m$-actions and properties of sets of differences, J. London Math. Soc. (2) 31 (1985), 295-304

[H-2002] N. Hegyvári, Note on difference sets in $\mathbb{Z}^n$, Periodica Math. Hung. 44 (2), 2002, 183-185

[Bo-1939] N. N. Bogolyubov, Some algebraical properties of almost periodes, (in Russian), Zapiski katedry matematichnoi fiziji (Kiev) 4 (1939), 185-194

[TV-2007] T. Tao & V. Vu, Additive Cominatorics, Cambridge Studies in Advanced Mathematics, 105. Cambridge University Press, Cambridge, 2006. xviii+512 pp

[H-2005] N. Hegyvári. On intersecting properties of partitions of integers, Combin. Probab. Comput. (14) 03, (2005), 319-323

[F1-1954] E. Følner, Generalization of a theorem of Bogoliuboff to topological Abelian groups. With an appendix on Banach mean values in non-Abelian groups, Math. Scandinavica 2 (1954), 5-18

[F2-1954)] E. Følner, Note on a generalization of a theorem of Bogoliuboff, Math. Scandinavica 2 (1954), 224-226

[HR-2007] N. Hegyvári & I. Z. Ruzsa, Additive Structure of Difference Sets and a Theorem of Flner, available at ; http://hegyvari.web.elte.hu/folner