ITERATED DIFFERENCE SETS IN σ-FINITE GROUPS

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Abstract. We improve on a previous result on iterated difference sets in arbitrary $\sigma-$finite groups.

1. Introduction

We investigate here the concept of iterated difference sets in the following way: for a given subset X of an arbitrary additively written group G, we define $D(X) = X - X = \{x - x' : x, x' \in X\}$ called difference set of X. We put $D_1 = D$, and for $k \geq 2$, $D_k(X) = D(D_{k-1}(X))$ for any $X \subseteq G$. In the case where G is the set of integers, Stewart and Tijdeman in [S-T] investigated the so-called iterated positive difference operation: for an infinite set A of positive integers, let $D^+(A)$ be the positive difference set defined by $D^+(A) = \{a - a' \mid a \geq a',\ a, a' \in A\}$. The k-fold iterated positive difference sequence $\{D_k^+(A);\ k \geq 0\}$ of A is defined by $D_0^+(A) = A$ and $D_k^+(A) = D^+(D_{k-1}^+(A))$ for $k \geq 1$. Stewart and Tijdeman observed that if a sequence A has positive upper density i.e.

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} > 0,$$

then the sequence $\{D_k^+(A);\ k \geq 0\}$ is stable i.e. there exists a k_0 such that, $D_{k+1}^+(A) = D_k^+(A)$ for every $k \geq k_0$.

We define the time of stability of A by $T(A) = \min\{k \mid D_{k+1}^+(A) = D_k^+(A)\}$. For instance, if $\overline{d}(A) > 1/2$, it is readily seen that $D^+(A)$ is the whole set of nonnegative integers, hence $T(A) \leq 1$. In [S-T] Stewart and Tijdeman gave an upper bound for $T(A)$ if the upper density of A is positive. They proved that if $0 < \overline{d}(A) \leq 1/2$ then $T(A) \leq 2\log_2(\overline{d}(A)^{-1})$, where $\log_2$ denotes the logarithmic function in base 2. This result was improved by Ruzsa in [Ru] where it is shown that under the same assumption on $\overline{d}(A)$, we have $T(A) \leq 2 + \log_2(\overline{d}(A)^{-1} - 1)$ and this bound is sharp.

At this point we note that a seemingly similar question is to consider the sequence $\{D_k(A);\ k \geq 1\}$ of the iterated difference sets without any restriction. The advantage of this question is that it can be handled in more general groups. Let G be a countable torsion group and let $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \subseteq \cdots$ be a sequence of finite subgroups of G. Then G is said to be σ-finite with respect to $\{H_n\}$ if $G = \bigcup_{n=1}^{\infty} H_n$.

We assume that G is a such group. Let $A \subseteq G$. The asymptotic upper density of A is defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap H_n|}{|H_n|}. \quad (1)$$

We introduce the time of stability in groups as well.

Assume the sequence $\{D_k(A)\,;\ k \geq 0\}$ is stable (i.e. for some k, $D_{k+1}(A) = D_k(A)$). Let $T(A, G)$ be the time of stability defined by

$$T(A, G) = \min\{k \mid D_{k+1}(A) = D_k(A)\}.$$

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In [He] the first named author extended the results of Stewart, Tijdeman and Ruzsa to σ-finite Abelian groups. He proved

Theorem A. *Let G be a σ-finite abelian group with respect to $\{H_n\}$ and let A be a non empty subset of G. Let $\overline{d}(A)$ be the upper density of A defined by (1). If $\overline{d}(A) > 0$, then*

$$T(A,G) \leq \log_2(\overline{d}(A)^{-1}) + 2.$$

It is worth mentioning that generalization to arbitrary linear operations (i.e. instead of $D(X)$, we consider operation $\Gamma(X) = aX - bX$) of this kind of problem is investigated in [H-H-P].

In the next section, we present some basic multiplicative results which are used in the rest of the section in order to show that Theorem A holds with a slightly worse bound without assuming G to be abelian.

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2. Iterated difference sets in finite group and σ-finite group

In this section, groups are not necessarily abelian and are written multiplicatively with identity element denoted by 1. Let G be any group and $A, A_1, A_2, \ldots, A_k$ be subsets of G. We denote by $A_1A_2\cdots A_k$ the subset of G of all products $x_1x_2\ldots x_k$ with $x_j \in A_j$, $j = 1\ldots, k$. We also define for $k \geq 1$, the k-fold product set $A^k = A\cdots A$ (k times), $A^{-1} = \{x^{-1} \mid x \in A\}$, and we put $D(A) = AA^{-1}$. We denote by $|A|$ the cardinality of A. Finally let $D_0(A) = A$, $D_1(A) = D(A)$ and $D_k(A) = D(D_{k-1}(A))$ for $k \geq 2$.

2.1. Preliminary results.

Lemma 2.1. *Let A and B be subsets of a finite group G such that $|A| + |B| \geq |G| + 1$. Then $AB = G$.*

Proof. Indeed if there is a $g \in G$ which is not in AB, then $A^{-1}g \cap B = \varnothing$ and so $|A^{-1}g| + |B| = |A| + |B| \leq |G|$, a contradiction. □

Lemma 2.2. *Let A be a generating subset of a finite group G such that $1 \in A$. For any non-empty subset X of G*

$$|XA| \geq \min\{|G|, |X| + |A|/2\}.$$

Proof. It is Theorem 1 in [Ol]. □

Lemma 2.3. *Let $A_1, A_2, \ldots, A_j$ be generating subsets of a finite group G such that $1 \in A_i$ for every i. If $A_1A_2\ldots A_j \neq G$, then*

$$|A_1| + \frac{|A_2| + \cdots + |A_{j-1}|}{2} + |A_j| \leq |G|.$$

Proof. Firstly remark that if $A_1A_2\ldots A_j \neq G$ we must have $A_1A_2\ldots A_i \neq G$, $i = 1, \ldots, j-1$, thus from Lemma 2.2 we deduce by induction that

$$|A_1A_2\ldots A_{j-1}| \geq |A_1| + \frac{|A_2| + \cdots + |A_{j-1}|}{2}.$$

Furthermore since $A_1A_2\ldots A_j \neq G$, by Lemma 2.1 we obtain $|A_1A_2\ldots A_{j-1}| + |A_j| \leq |G|$. Hence $|G| \geq |A_1A_2\ldots A_{j-1}| + |A_j| \geq |A_1| + \frac{|A_2|+\cdots+|A_{j-1}|}{2} + |A_j|$, which gives the required inequality. □

As a straightforward consequence of this result, we have

Lemma 2.4. *Let $j \geq 2$ be an integer and $A_1, A_2, \ldots, A_j$ be generating subsets of a finite group G such that $1 \in A_i$ for all i. If $j > 2|G|/|A| - 2$, then*

$$A_1 A_2 \ldots A_j = G.$$

2.2. **Results for finite and σ-finite groups.** We extend Theorem A to finite groups and σ-finite groups. We first consider the case of arbitrary finite groups:

Theorem 2.5. *Let A be a generating subset of a finite group G such that $1 \in A$. Let k_0 defined by*

$$k_0 = \begin{cases} 1 & \text{if } |G|/2 < |A| \leq |G|, \\ 2 & \text{if } |G|/3 < |A| \leq |G|/2, \\ \left\lfloor \log_2 \left(\frac{2|G|}{3|A|} - 1 \right) \right\rfloor + 3 & \text{if } |A| \leq |G|/3. \end{cases}$$

Then, for any integer $k \geq k_0$

$$D_k(A) = G. \tag{2}$$

Proof. If $|A| > |G|/2$ then by Lemma 2.1 with $B = A^{-1}$, we get $D_1(A) = G$. If $|A| > |G|/3$ then by Lemma 2.2 we get $|D_1(A)| > |G|/2$ and consequently $D_2(A) = G$ by Lemma 2.1. In the remaining of the proof, we assume that $|A| \leq |G|/3$. To see that (2) holds if $k \geq k_0$, we study the sequence $\{D_k(A)\,;\, k \geq 1\}$. We let $B = D(A)$. By Lemma 2.2, we have $|B| \geq 3|A|/2$ and $B^{-1} = B$ since $B = AA^{-1}$. We now observe by induction that for every integer $k \geq 1$, we have $B^{2^{k-1}} \subset D_{k-1}(B) = D_k(A)$. Let $k \geq k_0$. We thus have $j := 2^{k-1} > 2|G|/|B| - 2$. By Lemma 2.4 we get $B^j = G$ yielding $D_k(A) = D_{k-1}(B) = G$. □

This bounds allows us to improve that of [He, Proposition 3]. We obtain for k_0 defined in Theorem 2.5 that

$$T(A, G) \leq k_0 \tag{3}$$

for any subset A of an arbitrary finite group G.

We now show that Theorem A holds with a slightly worse bound for any non abelian σ-finite group as well. In the case where A is a subset of a σ-finite group G with upper density $\overline{d}(A)$ larger than $1/2$, it is readily seen that $A - A = G$, hence $T(A, G) \leq 1$. If $1/3 < \overline{d}(A) \leq 1/2$ then $|A \cap H_n| > |H_n|/3$ for infinitely many integers n. It follows from Lemma 2.2 that $|D(A) \cap H_n| > |H_n|/2$ for infinitely many n, yielding $\overline{d}(D(A)) > 1/2$. We conclude by the previous case that $D_2(A) = G$ hence $T(A, G) \leq 2$. We then formulate the remaining case:

Theorem 2.6. *Let G be a σ-finite group with respect to $\{H_n\}$ and let A be a non empty subset of G. Assume that A has a positive upper density and $\alpha := \overline{d}(A)^{-1} \geq 3$. Then*

$$T(A, G) \leq \lfloor \log_2(2\alpha/3 - 1) \rfloor + 3, \tag{4}$$

where $\lfloor u \rfloor$ denotes the greatest integer less than or equal to the real number u.

Proof. Since the function $\lfloor\ \rfloor$ is right-continuous, there exists a real number $0 < \eta < 1$ such that the right-hand side of (4) is equal to $k := \lfloor \log_2(2\alpha/3 - \eta) \rfloor + 3$. Let

$$\varepsilon := \min(-\log_2(\eta), 1 - \{\log_2(2\alpha/3 - \eta)\}) \tag{5}$$

where $\{u\} = u - \lfloor u \rfloor$ denotes the fractional part of u. Note that $\varepsilon > 0$ hence $2^\varepsilon > 1$, hence there exists an increasing sequence of integers $\{n_1 < n_2 < \cdots < n_i < \cdots\}$ such that

$$\overline{d}(A) < 2^\varepsilon \frac{|A \cap H_{n_i}|}{|H_{n_i}|}, \quad i \geq 1. \tag{6}$$

We claim that $T(A, G) \leq k$. Suppose that it is not the case. Thus

$$D_k(A) \neq D_{k+1}(A). \tag{7}$$

Let $A_n = A \cap H_n$. By (7) we infer that there exists an integer $n \in \{n_i\,;\, i \geq 1\}$ such that (6) holds and

$$D_k(A_n) \neq D_{k+1}(A_n). \tag{8}$$

Then by (5) and (6), we get

$$\frac{2\alpha}{3} - \eta > 2^{-\varepsilon}\frac{2|H_n|}{3|A_n|} - \eta \geq 2^{-\varepsilon}\left(\frac{2|H_n|}{3|A_n|} - 1\right),$$

hence

$$k \geq \log_2\left(\frac{2\alpha}{3} - \eta\right) + 2 + \varepsilon > \log_2\left(\frac{2|H_n|}{3|A_n|} - 1\right) + 2 + \varepsilon - \log_2(2^{\varepsilon}) = \log_2\left(\frac{2|H_n|}{3|A_n|} - 1\right) + 2.$$

By Theorem 2.5 and (3), we get $k \geq T(A_n, H_n)$, i.e. A_n is stable after k steps, a contradiction to (8). This ends the proof. □

References

[He] N. Hegyvári: On iterated difference sets in groups, *Period. Math. Hung.* **43** (2001), 105–110.

[H-H-P] N. Hegyvári, F. Hennecart and A. Plagne: Iterated linear operation on sets of positive upper density, (submitted 2007).

[Ol] J. E. Olson: On the Sum of Two Sets in a Group, *J. Number Th.* **18** (1984), 110–120.

[Ru] I. Z. Ruzsa: Iterated difference sets, *Studia Sci. Math. Hungar.* **22** (1987), 197–202.

[S-T] C. L. Stewart and R. Tijdeman: On density-difference sets of sets of integers, in: Studies in Pure Mathematics, p. 701–710, Birkhäuser Verlag, Basel, 1983.

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