Additive Iterations in Different Structures

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Let S be any structure and let $\mathcal{L}$ be any additive operation (strict definition later) and let $A \subseteq S$.

For

$$\mathcal{L}_1(A) := \mathcal{L}(A)$$

and

$$\mathcal{L}_{k+1}(A) := \mathcal{L}(\mathcal{L}_k(A))$$

the **orbit** of the sequence is

$$\mathcal{O}_{\mathcal{L},A} = \{\mathcal{L}_k(A) : k \in \mathbb{N}\}.$$

A typical question in Dynamical System:

WHAT CAN WE SAY ON THE ORBIT ?

History

$\mathcal{L}$ is the difference operation;

In the middle of the 1960's year, Erdős and Sárközy observed:

If A is a subset of integers with positive upper density,

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{A(n)}{n} > 0,$$

where

$$A(n) := \sum_{\substack{a\in A\\ 1\le a\le n}} 1.$$

The difference set

$$D(A) = \{a - a' : a, a' \in A\}.$$

has bounded gaps,

i.e. $\exists K > 0$, s.t. for

$$D(A) = \{d_1 < d_2 < \cdots < d_i < d_{i+1} < \dots\}$$
$$d_{i+1} - d_i < K$$

holds for $i = 1, 2, \dots$.

Remark:(Szemerédi, an easy exercise:) K does not depend on the density of A

Stewart and Tijdeman and later Ruzsa:

The orbit

$$\mathcal{O}_{\mathcal{D},A} \text{ is convergent or } \textit{stable}$$

provided A is a subset of integers with positive upper density, more precisely there exists a k_0 s.t.

$$D_{k+1} = D_k(A)$$

holds for $k \geq k_0$.

Define the time of stability of A by

$$T(A) = \min\{k : D^+_{k+1}(A) = D^+_k(A)\},$$

where the operation $D^+(\cdot)$ takes just the positive part of $D(\cdot)$.

Stewart and Tijdeman gave an estimation and later Ruzsa improved:

Theorem:

$$T(A) \leq 2 + \log_2(\overline{d}(A)^{-1} - 1).$$

Generalization; σ-finite Abelian groups

Let G be a countable torsion group,

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \subseteq \cdots$$

for each n, $H_n < G$.

G is said to be σ-finite with respect to $\{H_n\}$ if

$$G = \bigcup_{n=1}^{\infty} H_n.$$

For $A \subseteq G$, the asymptotic upper density of A is

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap H_n|}{|H_n|}.$$

The iterated difference set defined in the same way (without ordering), and the stability is too.

Theorem:[H]

Let G be a σ-finite Abelian group with respect to $\{H_n\}$ and let A be a subset of G for which $\overline{d}(A) > 0$. We have

$$T(A, G) \leq \log_2(\overline{d}(A)^{-1}) + 2.$$

For non-Abelian case:

Theorem ([Hennecart-H])
Let A be a generating subset of a finite group G such that $1 \in A$. Let k_0 defined by

$$k_0 = \begin{cases} 1 & \text{if } \frac{|G|}{2} < |A| \leq |G|, \\ 2 & \text{if } \frac{|G|}{3} < |A| \leq \frac{|G|}{2}, \\ \left\lfloor \log_2 \left(\frac{2|G|}{3|A|} - 1 \right) \right\rfloor + 3 & \text{if } |A| \leq \frac{|G|}{3}. \end{cases}$$

Then, for any integer $k \geq k_0$

$$D_k(A) = G.$$

For σ-finite groups:

Theorem ([Hennecart-H])

Let G be a σ-finite group with respect to $\{H_n\}$ and let A be a non empty subset of G. Assume that A has a positive upper density and $\alpha := \overline{d}(A)^{-1} \geq 3$. Then

$$T(A, G) \leq \lfloor \log_2(2\alpha/3 - 1) \rfloor + 3,$$

where $\lfloor u \rfloor$ denotes the greatest integer less than or equal to the real number u.

Further on the Difference Sets

V. Bergelson: additive structure of $D(A)$.

Theorem:([Bergelson])
There exists an infinite set B of integers for which

$$A - A \supseteq B + B + \cdots + B = B \cdot k,$$

provided A has positive upper density.

One can ensure $\overline{d}(B) > 0$, when $k = 2$.

For $k > 2$ only known

$$B(n) \to \infty,$$

when $n \to \infty$.

∘ Proof by ergodic way.

∘ Second proof: I observed a pure combinatorial proof (works on more general structure)

More on (Hypergraph) Ramsey-number ⇒ More on the size of B

∘ Third proof: With Ruzsa we use Følner theorem

Bohr set:

$$B(S,\varepsilon)=\{m\in\mathbb{Z}:\max_{s\in S}\|sm\|<\varepsilon\},$$

where S is a finite subset of the set of real numbers.

Theorem:[Følner]

Assume that $A\subseteq\mathbb{N}$ *and* $\overline{d}(A)>0$. *There exists a Bohr set* $B(S,\varepsilon)$ *such that*

$$B(S,\varepsilon)\setminus(A-A)$$

has density 0.

Theorem:[Ruzsa-H]

Let A be a set of integers, $\overline{\mathrm{d}}(A) > 0$. Let $f : \mathbb{N}_+ \mapsto \mathbb{N}_+$ be any function. There exists an infinite set C of integers, such that

$$A - A \supseteq FS(C) \cup FP(C),$$

where

$$FS(C) := \{ \sum_{c_i \in X} w(i) c_i : X \subseteq C, \\ |X| < \infty; w(i) \in \mathbb{N}, 1 \le w(i) \le f(i)\},$$

$$FP(C) := \{ \prod_{c_i \in X} c_i : X \subseteq C;\ |X| < \infty\}.$$

Further on the Difference Sets

A structure theorem of Hindman and Raimi:

Theorem:

There exists $E \subseteq \mathbb{N}$ *such that, whenever* $r \in \mathbb{N}$ *and* $\mathbb{N} = \bigcup_{i=1}^{r} D_i$ *there exist* $i \in \{1, 2, \ldots, r\}$ *and* $k \in \mathbb{N}$ *such that* $(D_i + k) \cap E$ *is infinite and* $(D_i + k) \setminus E$ *is infinite.*

One can perform it as

Theorem':

There exists $E \subseteq \mathbb{N}$ *such that, whenever* $r-$*coloring of integers, there exists a monochromatic subsets* D_i *and* $k \in \mathbb{N}$ *for which*

$$k \in (E - D_i) \cap (E^c - D_i),$$

and the representation of k *as a difference is infinite both in the two sets .(*E^c *is the complement of* E *with respect to* $\mathbb{N}$*.)*

A generalization

Theorem:[H]

Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ *be positive real numbers such that* $\sum_{i=1}^{r} \alpha_i = 1$.

There exists a disjoint partition $\mathbb{N} = \bigcup_{i=1}^{r} E_i$ *such that for every* $i \in \{1, 2, \ldots, r\}$,

$$d(E_i) = \alpha_i$$

and for each $t-$*coloring of integers, there exists a monochromatic subsets* F_m, *(*$m \in \{1, 2, \ldots, t\}$*) and an infinite sequence* $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ *such that for every*

$$h \in FS(\{x_n\}_{n=1}^{\infty}) \quad (*)$$

and every $i \in \{1, 2, \ldots, r\}$,

$$|(F_m + h) \cap E_i| = \infty.$$

Question:

Does there exist an infinite sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ such that instead of $(*)$ assume more; for every

$$h \in FS(\{x_n\}_{n=1}^{\infty}) \cup FP(\{x_n\}_{n=1}^{\infty})$$

and every $i \in \{1, 2, \ldots, r\}$,

$$|(F_m + h) \cap E_i| = \infty?$$

(A sharp ergodic-type result used to be like this. My combinatorial proof ensures the additive structure)

When the Operation $\mathcal{L}_{a,b}(A) = aA - bA$

(This section is a joint work with F. Hennecart and A. Plagne)

Let

$$\mathcal{L}_{a,b}(A) = aA - bA$$

Here generally the situation is different as in the difference operation;

Recall:

If A is a subset of integers

with positive upper density,

then $A - A$ has bounded gaps

Proposition:

Let $X \subset \mathbb{N}$ and $a, b \in \mathbb{N}$, such that $a \geq b \geq 1$ and $\overline{\mathrm{d}}(X) > a/(a+1)$. Then the gaps in both sets $\Gamma_{a,b}(X) = aX - bX$ and $\Gamma_{b,a}(X) = bX - aX$ are bounded from above by a.

On the orbit

$$\mathcal{O}_{\mathcal{L},A} = \{\mathcal{L}_k(A) : k \in \mathbb{N}\}.$$

Example 1:

Let

$$\mathcal{L}_{a,b}(A) = A - 2A; \qquad A := 1 + 3\mathbb{Z}.$$

Then

$$\mathcal{L}_{2k-1}(A) = 1 + A; \qquad \mathcal{L}_{2k}(A) = A.$$

(not a surprise) the orbit is **not stable**.

Modify the notion of the orbit

Convolution of linear operations

For operations $\mathcal{L}_{a_1,b_1}, \mathcal{L}_{a_2,b_2}$ let

$$\mathcal{L}_{a_1,b_1} \circ \mathcal{L}_{a_2,b_2}(X) = \mathcal{L}_{a_1,b_1}(\mathcal{L}_{a_2,b_2}(X)) =$$

$$= a_1a_2X + b_1b_2X - a_1b_2X - a_2b_1X.$$

and generally for $\Omega = (\{a_1, b_1), (a_2, b_2), \ldots (a_s, b_s)\}$ a finite sequence of couples of positive integers

$$Conv_{j=1}^{s}\mathcal{L}_{a_j,b_j}(X) = Conv_{j=1}^{s-1}\mathcal{L}_{a_j,b_j} \circ \mathcal{L}_{a_s,b_s}(X)$$

Now the orbit:

$$\mathcal{O}_\Omega = \{X \cup Conv_{j=1}^{s}\mathcal{L}_{a_j,b_j}(X) : s \in \mathbb{N}\}.$$

May be the orbit is eventually periodically stable

(that from some point $\mathcal{L}_{k+p}(A) = \mathcal{L}_k(A)$ for some $p \geq 1$ and any large enough k).

Example 2:

Let again

$$A = 1 + 3\mathbb{Z}; \quad \alpha \in (0, 1)$$

$\alpha = 0.\alpha_1\alpha_2\ldots$ (dyadic expansion) is an irrational.

Put $(a_i, b_i) = (2, 1)$ if $\alpha_i = 0$ and $(a_i, b_i) = (3, 1)$ otherwise.

Then

$$\mathcal{L}_k(A) = A$$

if $\alpha_k = 0$ and

$$\mathcal{L}_k(A) = -A$$

otherwise.

Hence a good definition is: the orbit is $t-$stable respect to Ω, if

$$|\mathcal{O}_\Omega| \leq t.$$

Which condition does the t-stability ensure?

Only a density condition not:

Theorem:

Let $L \geq 1$ be an integer and $(a_j, b_j)_{j\in\mathbb{N}}$, be a sequence of positive integers such that $|a_j|, |b_j| \leq L$ for any $j \geq 1$. Then for any positive integer t, there exists a set $A \subset \mathbb{N}$ with asymptotic density

$$d(A) \geq (2L)^{-t}$$

such that A is not t-stable.

The main result is:

Theorem:

Let $L \geq 2$ *be an integer,* A *be an increasing sequence of integers and assume that* $1/\beta = \overline{d}(A) > 0$. *Let* $(a_j, b_j)_{j\in\mathbb{N}}$ *be a sequence of couples of positive integers such that* $a_j \leq L$, $b_j \leq L$, $\gcd(a_j, b_j) = 1$ *for any* $j \geq 1$. *and*

$$K = \lfloor c(\log\beta \log L + L) \rfloor$$

be a positive integer and c *is a sufficiently large absolute constant. Then*

(i) *there exists a modulus* g *satisfying*

$$g \leq L^{K+1}$$

such that for any $k \geq K$, $\mathcal{L}_k(A)$ *is fully periodic modulo* g.

(ii) *the orbit* $\mathcal{O}_\Omega$ *is* $(K + g^3 L^2)$*-stable.*